Numerical Analysis2

Chapter Two

Numerical Integration

2.1 Introduction

Numerical integration is a primary tool used by engineers and scientists to obtain approximate answers for definite integrals that cannot be solved analytically**.** The numerical evaluation of integrals for the differentiable function f(x) in the interval [a,b] is defined as follows:

$$I[f] = \int_a^b f(x)dx \tag{1}$$

where f(x) ϵ C[a,b], for example

$$\int_0^5 e^{-x^2} dx$$

cannot be found analytically. The goal is to approximate the definite integral of *f (x)* over the interval [*a*, *b*] by evaluating *f (x)* at a finite number of sample points.

Definition: Suppose that $a = x_0 < x_1 < \cdot \cdot \cdot < x_m = b$. A formula of the form

$$Q[f] = \sum_{k=0}^{m} w_k f(x_k) = w_0 f(x_0) + w_1 f(x_1) + \cdots + w_m f(x_m) \tag{2}$$

with the property that

$$I[f] = \int_a^b f(x)dx = Q[f] + E[f] \tag{3}$$

is called a numerical integration or ***quadrature*** formula. The term $E[f]$ is called the ***truncation error*** for integration. The values $\{x_k\}_{k=0}^{m}$ are called the ***quadrature nodes***, and $\{w_k\}_{k=0}^{m}$ are called the ***weights***.

2.2 Trapezoidal Method

The derivative of simple Trapezoidal rule formula to approximate value of the integral in eq.(1) we approximate the function f(x) as Newton forward interpolation formula and then integration over the interval $[x_0,x_1]$, where

$x=x_m=x_0+mh$, m=o,1,2,…, n

and $x_0=a$, $x_m=b$, $h=\frac{b-a}{n}$

Now, let **n=1**, we have

$$\int_{a=x_0}^{b=x_1} f(x)dx=\int_0^1 f(x_m)dx\frac{dm}{dm}=\int_0^1 f(x_m)dm.h=h\int_0^1 f(x_m)dm$$

$$=h\int_0^1[f_0+m\Delta f_0+\frac{m(m-1)}{2!}\Delta^2 f_0+\frac{m(m-1)(m-2)}{3!}\Delta^3 f_0+\cdots]dm$$

$$=h\int_0^1[f_0+m\Delta f_0+(\frac{m^2}{2}-\frac{m}{2})\Delta^2 f_0+(\frac{m^3}{6}-\frac{m^2}{2}+\frac{m}{3})\ \ \Delta^3 f_0+\cdots]dm$$

$$=h\left[mf_0+\frac{m^2}{2}\Delta f_0+(\frac{m^3}{6}-\frac{m^2}{4})\Delta^2 f_0+(\frac{m^4}{24}-\frac{m^3}{6}+\frac{m^2}{6})\ \ \Delta^3 f_0+\cdots\right]_0^1$$

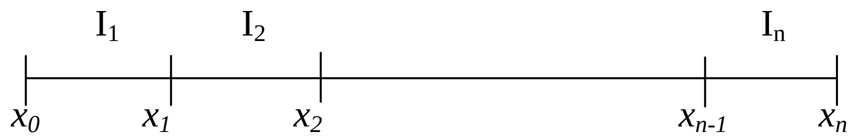
$$=h\left[f_0+\frac{1}{2}\Delta f_0-\frac{1}{12}\Delta^2 f_0+\frac{1}{24}\Delta^3 f_0+\cdots\right]$$

$$\int_{a=x_0}^{b=x_1} f(x)dx\cong\frac{h}{2}[f_0+f_1] \qquad (4)$$

Equation (4) is called **Simple Trapezoidal Rule formula** ,the local truncation error in equation (4) is

$$\mathrm{L.T.E.}=\frac{-h^3}{12}f''(\theta)\ \ ,\ \ \theta\epsilon(x_0,x_1)$$

Now, let **n >1** we have subintervals $[x_{i-1}, x_i]$ and we apply eq.(4) at each subinterval we get:



$$\int_{a=x_0}^{b=x_n} f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx$$

$$= \frac{h}{2}[f_0 + f_1] + \frac{h}{2}[f_1 + f_2] + \cdots + \frac{h}{2}[f_{n-1} + f_n]$$

$$= \frac{h}{2}[f_0 + 2(f_1 + f_2 + \cdots + f_{n-1}) + f_n] \qquad (5)$$

Equation (5) is called **Composite Trapezoidal Rule formula** ,and the total truncation error in equation (5) is

$$\mathrm{E_T} = \left[-\frac{h^3}{12} f''(\theta_1)\right] + \left[-\frac{h^3}{12} f''(\theta_2)\right] + \cdots + \left[-\frac{h^3}{12} f''(\theta_n)\right]$$

Where $\theta_i \epsilon (x_{i-1}, x_i)$, $i = 1,2, \dots, n$

$$\mathrm{E_T} = -\mathrm{n}\left[\frac{h^3}{12} f''(\theta)\right] = -\frac{(b-a)^3}{12n^2} f''(\theta) \quad , \quad \theta \epsilon (a,b)$$

This is **Composite error term** in Composite Trapezoidal Rule.

Example(1): Use Trapezoidal rule to evaluate the integral

$$\int_0^1 \frac{1}{1+x^2} dx$$

Considering h=0.25

Solution:

a=0 , b=1 , $f(x) = \frac{1}{1+x^2}$ and h=0.25

$x_i = x_{i-1} + h \Longrightarrow$

$x_0=0$, $f(x_0)=f(0)=1$

$x_1=0+0.25=0.25$, $f(x_1)=f(0.25)=0.9412$

$x_2=0.25+0.25=0.5$, $f(x_2)=f(0.5)=0.8$

$x_3=0.5+0.25=0.75$, $f(x_3)=f(0.75)=0.64$

$x_4=0.75+0.25=1$, $f(x_4)=f(1)=0.5$

$$I[f] = \int_0^1 \frac{1}{1+x^2}dx = \frac{h}{2}[f_0 + 2(f_1 + f_2 + f_3) + f_4]$$

$$= \frac{0.25}{2}\ [1 + 2(\ 0.9412 + 0.8 + 0.64) + 0.5]$$

$$= 0.7828$$

Example(2): Use Trapezoidal rule to evaluate the integral

$$\int_0^1 (x^3+1)dx$$

Considering n=1 and n=4.

Solution:

a=0 , b=1 , $f(x) = (x^3+1)$ and n=1 $\Longrightarrow h = \frac{b-a}{n} = 1$

$x_i = x_{i-1} + h \Longrightarrow$