Bivariate Normal distribution:-

① If X_1 and X_2 are normally distribution then the bivariate density function is

$$f(x_1, x_2) = \frac{1}{(2\pi)\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]}$$

Proof

$$f(\underline{X}) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}\left\{(\underline{X}-\underline{\mu})'\Sigma^{-1}(\underline{X}-\underline{\mu})\right\}}$$

$$\underline{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad \underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ & \sigma_2^2 \end{bmatrix}$$

$$\rho = \frac{\sigma_{12}}{\sigma_1\sigma_2} \Longrightarrow \sigma_{12} = \rho\sigma_1\sigma_2$$

$$|\Sigma| = \sigma_1^2\sigma_2^2 - \rho^2\sigma_1^2\sigma_2^2 \Rightarrow \sigma_1^2\sigma_2^2(1-\rho^2)$$

$$\Sigma^{-1} = \frac{\operatorname{adj}\Sigma}{|\Sigma|} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}\begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_2\sigma_1 & \sigma_1^2 \end{bmatrix}$$

and $|\Sigma|^{\frac{1}{2}} = \sqrt{\sigma_1^2\sigma_2^2(1-\rho^2)} = \sigma_1\sigma_2\sqrt{(1-\rho^2)}$

$$f(x_1, x_2) = \frac{1}{(2\pi)\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2}\begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}' \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}\begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}\begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}}$$

The p.d.f defined in eq (1) is called the bivariate normal distribution when x_1, x_2 are correlated. This distribution is indicated in symbols as :-

$$\underline{X} \sim N_2(\underline{\mu}, \Sigma)$$

or $\underline{X} \sim N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho_{12})$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2\sigma_1^2\sigma_2^2(1-\rho^2)}\left[\sigma_2^2(x_1-\mu_1)^2 - 2\rho\sigma_1\sigma_2(x_1-\mu_1)(x_2-\mu_2) + \sigma_1^2(x_2-\mu_2)^2\right]\right\}$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right]\right\} \quad \text{--- (4)}$$

ملاحظة:
تمثل دالة كثافة احتمال التوزيع الطبيعي الثنائي المتغير عندما تكون المكونات العشوائية ($\underline{X}$) غير مستقلة

Notes

(1) If $\rho = 0$ that is mean x_1 & x_2 ar independent, therefore

$$f(\underline{X}) = \frac{1}{2\pi\sigma_1\sigma_2}\exp\left[-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)\right]$$

$$= \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2}\right)} \cdot e^{-\frac{1}{2}\left(\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)}$$

$$f(\underline{X}) = \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2}\right)} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{1}{2}\left(\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)}$$

$$\therefore f(x_1, x_2) = f(x_1) \cdot f(x_2)$$

(2) If $\rho = 0$ that is mean X_1 & X_2 are independent therefore :-

$$f(x_1, x_2) = \frac{1}{(2\pi)\,\sigma_1 \sigma_2} \exp\left\{-\frac{1}{2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1} \exp\left\{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2\right\} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_2} \exp\left\{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right\}$$

$$= f(x_1) \,.\, f(x_2)$$

$\rho=-1,\ \rho=1$

(3) If X_1 & X_2 are Perfectly Correlated (i.e: $\rho = \mp 1$), then Σ is Singular matrix (i.e. $|\Sigma| = 0$), in this case we can't find the joint density function

$$\Rightarrow \rho = \frac{\sigma_{12}}{\sigma_1 \sigma_2} \Rightarrow \mp 1 = \frac{\sigma_{12}}{\sigma_1 \sigma_2} \Rightarrow \sigma_{12} = \mp \sigma_1 \sigma_2$$

$$|\Sigma| = \begin{vmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ & \sigma_2^2 \end{vmatrix} = \sigma_1^2 \sigma_2^2 - \rho^2 \sigma_1^2 \sigma_2^2$$

$$= \sigma_1^2 \sigma_2^2 (1 - \rho^2)$$

$$= \sigma_1^2 \sigma_2^2 (1 - (\mp 1)^2) = 0$$

$$\therefore |\Sigma| = 0$$

In general, $|\Sigma| > 0$ then $-1 < \rho < 1$

Proof:

$|\Sigma| > 0 \Rightarrow \sigma_1^2 \sigma_2^2 (1 - \rho^2) > 0$

$\sigma_1^2 \sigma_2^2 - \sigma_1^2 \sigma_2^2 \rho^2 > 0$

$\sigma_1^2 \sigma_2^2 > \sigma_1^2 \sigma_2^2 \rho^2$

$\Rightarrow 1 > \rho^2$

$\Rightarrow -1 < \rho < 1$

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Ex If $\underline{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, $\underline{\mu} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

$\Sigma = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$

and X_1 , X_2 are normally distribution.

(1) Write the joint density function in the simplest form. ابسط صورة

(2) Find the Correlation Coefficient ρ

Sol

$$f(x_1, x_2) = \frac{1}{(2\pi)\, |\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}\left\{(\underline{X}-\underline{\mu})' \Sigma^{-1} (X-\mu)\right\}}$$

$$(\underline{X}-\underline{\mu})' \Sigma^{-1} (\underline{X}-\mu) = (x_1+2 \quad x_2-4)\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}^{-1} \begin{bmatrix} x_1+2 \\ x_2-4 \end{bmatrix}$$

$$|\Sigma| = 4-1 = 3$$

$$\Sigma^{-1} = \frac{1}{3}\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 4/3 & -1/3 \\ -1/3 & 1/3 \end{pmatrix}$$

$$\therefore (\underline{X}-\underline{\mu})' \Sigma^{-1} (X-\mu) = \underset{1\times 2}{(x_1+2 \quad x_2-4)} \underset{2\times 2}{\begin{pmatrix} 4/3 & -1/3 \\ -1/3 & 1/3 \end{pmatrix}} \underset{2\times 1}{\begin{pmatrix} x_1+2 \\ x_2-4 \end{pmatrix}}$$

$$= \underset{1\times 2}{\left(\frac{4x_1}{3} + \frac{8}{3} - \frac{4x_2}{3} + \frac{16}{3} \quad -\frac{x_1}{3} - \frac{2}{3} + \frac{x_2}{3} - \frac{4}{3}\right)} \underset{2\times 1}{\begin{bmatrix} x_1+2 \\ x_2-4 \end{bmatrix}}$$

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(2) To find ρ :

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$$

$\sigma_1^2 = 1 \Longrightarrow \sigma_1 = 1$

$\sigma_2^2 = 4 \Longrightarrow \sigma_2 = 2$

$\rho\,\sigma_1\sigma_2 = 1 \Longrightarrow \rho(1)(2) = 1 \Longrightarrow 2\rho = 1$

$\therefore \rho = \frac{1}{2}$

ex Let $\underline{X} = (X_1 \;\; X_2)' \sim N_2(0, 0, \sigma_1^2, \sigma_2^2, \rho_{12})$

show that the random variable

$U_1 = \frac{X_1}{\sigma_1} + \frac{X_2}{\sigma_2}$ and $U_2 = \frac{X_1}{\sigma_1} - \frac{X_2}{\sigma_2}$

are independent with $E(U_1) = E(U_2) = 0$

and $var(U_1) = 2(1+\rho)$, $V(U_2) = 2(1-\rho)$

Solu.:

$E(U_1) = \frac{E(X_1)}{\sigma_1} + \frac{E(X_2)}{\sigma_2} = 0$

$E(U_2) = \frac{E(X_1)}{\sigma_1} - \frac{E(X_2)}{\sigma_2} = 0$

$$v(U_1) = \frac{v(X_1)}{\sigma_1^2} + \frac{v(X_2)}{\sigma_2^2} + \frac{2\,Cov(X_1, X_2)}{\sigma_1 \sigma_2}$$

$$v(U_1) = \frac{\sigma_1^2}{\sigma_1^2} + \frac{\sigma_2^2}{\sigma_2^2} + \frac{2\rho\,\sigma_1\sigma_2}{\sigma_1\sigma_2}$$

$$= 1 + 1 + 2\rho = 2(1+\rho)$$

$$v(U_2) = \frac{v(X_1)}{\sigma_1^2} + \frac{v(X_2)}{\sigma_2^2} - \frac{2\,Cov(X_1, X_2)}{\sigma_1\sigma_2}$$

$$= 1 + 1 - \frac{2\rho\,\sigma_1\sigma_2}{\sigma_1\sigma_2}$$

$$= 2(1-\rho)$$

$$Cov(U_1, U_2) \overset{?}{=} 0$$

$$Cov(U_1, U_2) = E(U_1U_2) - \overset{0}{E(U_1)}\,\overset{0}{E(U_2)} = E(U_1U_2)$$

$$= E\left[\frac{X_1}{\sigma_1} + \frac{X_2}{\sigma_2}\right]\left[\frac{X_1}{\sigma_1} - \frac{X_2}{\sigma_2}\right]$$

$$= E\left[\frac{X_1^2}{\sigma_1^2} - \frac{X_2^2}{\sigma_2^2}\right]$$

$$= \frac{\sigma_1^2}{\sigma_1^2} - \frac{\sigma_2^2}{\sigma_2^2} = 0$$

$v(x) = Ex^2 - (Ex)^2$

$\therefore Ex^2 = v(x) + (Ex)^2$

Thus U_1 and U_2 are independent