



**Example (9):**

Find the root of the equation  $x^3 - 3x + 2 = 0$  with  $x_0 = -2.4$ , and  $\varepsilon = 5 \times 10^{-5}$  by using Newton-Raphson method.

**Solution:**

Since  $f(x) = x^3 - 3x + 2$  then  $f'(x) = 3x^2 - 3$  and  $x_0 = -2.4$ , use the formula

$$x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})}, \quad i = 1, 2, 3, \dots$$

$$\begin{aligned} \mathbf{i=1} \longrightarrow x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = -2.4 - \frac{f(-2.4)}{f'(-2.4)} \\ &= -2.4 - \frac{(-2.4)^3 - 3(-2.4) + 2}{3(-2.4)^2 - 3} = -2.0763 \end{aligned}$$

$$|x_1 - x_0| = |-2.0763 - (-2.4)| = |0.3237| > \varepsilon = 5 \times 10^{-5}$$

$$\longrightarrow x_0 = x_1 = -2.0763$$

$$\begin{aligned} \mathbf{i=2} \longrightarrow x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = -2.0763 - \frac{f(-2.0763)}{f'(-2.0763)} \\ &= -2.003699 \end{aligned}$$

$$|x_2 - x_1| = |-2.003699 - (-2.0763)| > \varepsilon = 5 \times 10^{-5}$$

$$\longrightarrow x_1 = x_2 = -2.003699$$

$$\begin{aligned} \mathbf{i=3} \longrightarrow x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = -2.003699 - \frac{f(-2.003699)}{f'(-2.003699)} \\ &= -2.000009 \end{aligned}$$

$$|x_3 - x_2| = |-2.000009 - (-2.003699)| = 0.003690 > \varepsilon = 5 \times 10^{-5}$$

$$\longrightarrow x_2 = x_3 = -2.000009$$

$$\mathbf{i=4} \longrightarrow x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = -2$$

$$|x_4 - x_3| = |-2 - (-2.000009)| = 0.000009 < \varepsilon = 5 \times 10^{-5}$$

The iteration is stop with the root  $x_4 = -2$

### Algorithm (Newton-Raphson method)

Input:  $x_0, \varepsilon, f(x)$

Step(1): Set  $i=1$

Step(2): Compute  $x = x_0 - \frac{f(x_0)}{f'(x_0)}$

Step(3): If  $|x - x_0| < \varepsilon$  then print  $x$  is a root and stop.

Step(4): Else if set  $x_0 = x$  and  $i=i+1$  then go to step(2).

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### Special cases of Newton-Raphson method

#### 1- Find the square roots (i.e. $\sqrt{a}$ ):

Let  $f(x)=x^2-a$

If  $f(x)=0$  then  $x = \sqrt{a}$

By using Newton-Raphson formula in equation(3) we have

$$\begin{aligned}x_i &= x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})} \\&= x_{i-1} - \frac{x_{i-1}^2 - a}{2x_{i-1}} \\&= \frac{2x_{i-1}^2 - x_{i-1}^2 + a}{2x_{i-1}} \\&= \frac{1}{2} \left( x_{i-1} + \frac{a}{x_{i-1}} \right), \quad i = 1, 2, 3, \dots\end{aligned}$$

**Example (10):** Find the square root of the number 5 (i.e.  $\sqrt{5}$ ), take  $\varepsilon = 10^{-6}$  by using Newton-Raphson method.

**Solution:** Set  $f(x) = x^2 - 5$  and  $f'(x) = 2x$ , take  $x_0=2$

Now by using Newton-Raphson formula we have:

$$x_i = x_{i-1} - \frac{x_{i-1}^2 - 5}{2x_{i-1}}$$

$$= \frac{1}{2} \left( x_{i-1} + \frac{5}{x_{i-1}} \right)$$

**i=1**  $\longrightarrow$   $x_1 = \frac{1}{2} \left( x_0 + \frac{5}{x_0} \right) = \frac{1}{2} \left( 2 + \frac{5}{2} \right) = 2.25$

$$|x_1 - x_0| > \varepsilon = 10^{-6}, \quad x_0 = x_1$$

**i=2**  $\longrightarrow$   $x_2 = \frac{1}{2} \left( x_1 + \frac{5}{x_1} \right) = 2.236111$

$$|x_2 - x_1| > \varepsilon = 10^{-6}, \quad x_1 = x_2$$

**i=3**  $\longrightarrow$   $x_3 = \frac{1}{2} \left( x_2 + \frac{5}{x_2} \right) = 2.236068$

$$|x_3 - x_2| > \varepsilon = 10^{-6}, \quad x_2 = x_3$$

**i=4**  $\longrightarrow$   $x_4 = \frac{1}{2} \left( x_3 + \frac{5}{x_3} \right) = 2.236068$

$$|x_4 - x_3| < \varepsilon = 10^{-6}$$

stop with  $x_4=2.236068$  is the root.

**2- Find the  $k^{\text{th}}$  root for  $a>0$  (i.e.  $\sqrt[k]{a}$ ) :**

$$\text{Let } f(x) = x^k - a \Rightarrow f'(x) = kx^{k-1}$$

If  $f(x)=0$  then  $x = \sqrt[k]{a}$ , By using Newton-Raphson formula in equation(3) we have

$$x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})} = x_{i-1} - \frac{x_{i-1}^k - a}{kx_{i-1}^{k-1}} = \left( 1 - \frac{1}{k} \right) x_{i-1} + \frac{a}{k} x_{i-1}^{1-k} \quad ; i=1, 2, \dots$$

**Example (11):** Find  $\sqrt[3]{2}$ , take  $\varepsilon = 10^{-4}$  by using Newton-Raphson method.

**Solution:** Set  $f(x) = x^3 - 2$  and  $f'(x) = 3x^2$ , take  $x_0 = 1.2$

Now by using Newton-Raphson formula we have:

$$x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})} = x_{i-1} - \frac{x_{i-1}^3 - 2}{3x_{i-1}^2} = \left(1 - \frac{1}{3}\right)x_{i-1} + \frac{2}{3}x_{i-1}^{-2}$$

$$x_0 = 1.2, x_1 = 1.26296, x_2 = \quad, x_3 = \quad, x_4 = \quad$$

**3- Find  $\frac{1}{a}$ ,  $a \neq 0$  without using division :**

Let  $f(x) = \frac{1}{x} - a$ ,  $f'(x) = -\frac{1}{x^2}$ . If  $f(x) = 0$  then  $x = \frac{1}{a}$

Now by using Newton-Raphson method we have:

$$x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})} = x_{i-1} - \frac{\frac{1}{x_{i-1}} - a}{-\frac{1}{x_{i-1}^2}} = x_{i-1} + x_{i-1} - ax_{i-1}^2 = (2 - ax_{i-1})x_{i-1}, \quad i=1, 2, \dots$$

**Example (12):** Use Newton-Raphson method to find  $\frac{1}{4}$ , take  $x_0 = 0.15$ ,  $\varepsilon = 10^{-4}$ .

**Solution:** Let  $f(x) = \frac{1}{x} - 4$ ,  $f'(x) = -\frac{1}{x^2}$ .

$$x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})} = (2 - 4x_{i-1})x_{i-1}, \quad i=1, 2, \dots$$

$$x_1 = (2 - 4x_0)x_0 = 0.21$$

$$|x_1 - x_0| > \varepsilon = 10^{-4}, \quad x_0 = x_1$$

$$x_2 = (2 - 2x_1)x_1 = 0.2436$$

$$x_3 = 0.24984, \quad x_4 = 0.24999, \quad x_5 = 0.249996$$

$$\text{stop since } |x_5 - x_4| \leq \varepsilon = 10^{-4}.$$

### Home work:

Find an approximate root of equation  $f(x)=x \sin(x)-1=0$  in the interval  $[0,2]$  by using Newton-Raphson method with  $x_0=1$ , and  $\varepsilon = 10^{-5}$ .

### Properties of the Newton-Raphson method

### خواص الطريقة

- 1- سرعة الاقتراب في طريقة Newton-Raphson تربيعية (Quadratic convergent) .
- 2- تتطلب حساب مشتقة الدالة عند كل تكرار ولا يمكن ضمان التقارب عندما  $f'(x_i) \approx 0$  .
- 3- نوع الاقتراب محلي (Locally convergent) لان نقطة البداية يجب ان تكون قريبة من الجذر الحقيقي.

4- صيغة الخطأ في طريقة نيوتن رافسون هي  $e_{i+1} \leq \frac{M}{2m} e_i^2$  حيث ان  $|f''(x)| \leq M$  ,  $m \leq |f'(x)|$

لكل قيم  $x$  في الفترة التي تحتوي  $x_i, P$  .