2.3 Simpson's One-Third Method

The derivative of simple Simpson's One-Third rule formula to approximate value of the integral in eq.(1) we approximate the function f(x) as Newton forward interpolation formula and then integration over the interval $[x_0,x_2]$, where

$x=x_m=x_0+mh$, $m=0,1,2,\ldots, n$

Now, let **n=2**, we have

$$\int_{a=x_0}^{b=x_2} f(x)dx = \int_0^2 f(x_m)dx\frac{dm}{dm} = \int_0^2 f(x_m)dm.h = h\int_0^2 f(x_m)dm$$

$$=h\int_0^2[f_0 + m\Delta f_0 + \frac{m(m-1)}{2!}\Delta^2 f_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 f_0 + \cdots]dm$$

$$= h\left[mf_0 + \frac{m^2}{2}\Delta f_0 + (\frac{m^3}{6} - \frac{m^2}{4})\Delta^2 f_0 + (\frac{m^4}{24} - \frac{m^3}{6} + \frac{m^2}{6})\ \ \Delta^3 f_0 + \cdots\right]_0^2$$

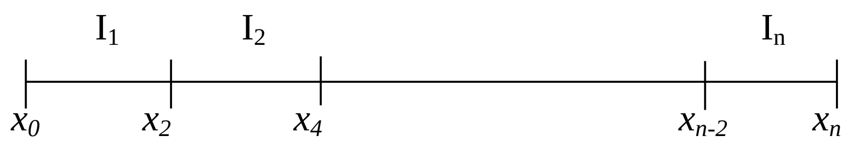
$$= h\left[2f_0 + 2\Delta f_0 + \frac{1}{3}\Delta^2 f_0 + \cdots - 0\right]$$

$$\int_{a=x_0}^{b=x_2} f(x)dx \cong \frac{h}{3}[f_0 + 4f_1 + f_2] \qquad (6)$$

Equation (6) is called **Simple Simpson's One-Third Rule formula** ,the local truncation error in equation (6) is

$$\text{L.T.E.} = \frac{-h^5}{90} f^4(\theta) \ \ , \ \ \theta\epsilon(x_0, x_2)$$

In general, we have subintervals $[x_{i-2}, x_i]$ and we apply eq.(6) at each subinterval we get:



$$\int_{x_i}^{x_{i+2}} f(x)dx \cong \frac{h}{3}[f_i + 4f_{i+1} + f_{i+2}]$$

then

$$\int_{a=x_0}^{b=x_n} f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)dx$$

$$= \frac{h}{3}[f_0 + 4f_1 + f_2] + \frac{h}{3}[f_2 + 4f_3 + f_4] + \cdots + \frac{h}{3}[f_{n-2} + 4f_{n-1} + f_n]$$

$$= \frac{h}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \cdots + 4f_{n-1} + f_n\right]$$

$$= \frac{h}{3}[f_0 + 4\sum_{i=1}^{n} f(x_{2i-1}) + 2\sum_{i=1}^{n-1} f(x_{2i}) + f_n] \qquad (7)$$

Equation (7) is called **Composite Simpson's One-Third Rule formula** ,and the total truncation error in equation (7) is

$$E_s = \left[\frac{-h^5}{90} f^4(\theta_1)\right] + \left[\frac{-h^5}{90} f^4(\theta_2)\right] + \cdots + \left[-\frac{h^5}{90} f^4(\theta_n)\right]$$

$$E_s = -n\left[\frac{-h^5}{180} f^4(\theta\quad)\right] = -\frac{(b-a)^5}{180n^4} f^4(\theta) \quad , \quad \theta\epsilon(a,b)$$

This is **Composite error term** in Composite Simpson's One-Third Rule.

Example(4): Evaluate the following integral using Simpson's One-Third Rule taking h=0.25.

$$\int_0^1 \frac{1}{1+x^2} dx$$

Solution:

a=0 , b=1 , $f(x) = \frac{1}{1+x^2}$ and h=0.25 , $n = \frac{b-a}{h} = 4$

$x_i = x_{i-1} + h$ ⟹

$x_0=0$, $f(x_0)=f(0)=1$

$x_1=0+0.25=0.25$, $f(x_1)=f(0.25)=0.9412$

$x_2=0.25+0.25=0.5$, $f(x_2)=f(0.5)=0.8$

$x_3=0.5+0.25=0.75$, $f(x_3)=f(0.75)=0.64$

$x_4=0.75+0.25=1$, $f(x_4)=f(1)=0.5$

$$I[f] = \int_0^1 \frac{1}{1+x^2}dx == \frac{h}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \cdots + 4f_{n-1} + f_n\right]$$

$$= \frac{0.25}{3}[1 + 4(0.9412) + 2(0.8) + 4(0.64) + 0.5]$$

$$= 0.7854$$

Example(5): Use Simpson's One-Third Rule to evaluate the integral

$$\int_0^1 (x^3 + 1)dx$$

Considering n=2.

Solution:

a=0 , b=1 , $f(x) = (x^3 + 1)$ and

n=2 ⟹ $h = \frac{b-a}{n} = \frac{1}{2}$

$x_i = x_{i-1} + h \quad \Longrightarrow$

$x_0 = 0 \quad , \quad f(x_0) = f(0) = 1$

$x_1 = 0 + 1/2 = 1/2 \quad , \quad f(x_1) = f(1/2) = 9/8$

$x_2 = 1/2 + 1/2 = 1 \quad , \quad f(x_2) = f(1) = 2$

$$I[f] = \int_0^1 (x^3 + 1)dx = \frac{h}{3}[f_0 + 4f_1 + f_2]$$

$$= \frac{1/2}{3}\left[1 + 4\left(\frac{9}{8}\right) + 2\right] = \frac{12}{15} = 1.25$$

Evaluation the number of subintervals in Simpson's One-Third rule :

From the error formula

$$E_S = -\frac{(b-a)^5}{180n^4} f^4(\theta) \quad , \quad \theta \epsilon (a, b)$$

Let $|f^4(\theta)| \leq M$ then the error E_S for the composite Simpson's One-Third rule is less than accuracy ϵ

$$|E_S| = \left|-\frac{(b-a)^5}{180n^4} f^4(\theta)\right| \leq \epsilon$$

$$\Longrightarrow \quad \frac{(b-a)^5}{180n^4} M \leq \epsilon$$

$$\Longrightarrow \quad n \geq \sqrt[4]{\frac{(b-a)^5}{180\epsilon} M}$$

Example(6): Find the value of the integral with accuracy $\in = 0.001$ by using Simpson's One-Third rule

$$I(f) = \int_{0.5}^{1} \cos x \, dx$$

Solution:

a=0.5 , b=1 , $f(x) = \cos x$ and $\in = 0.001$

$$I[f] = \int_{0.5}^{1} \cos x \, dx \cong 0.36206$$

Algorithm of Simpson's One-Third rule:

Input: a , b , n , f(x)

Step(1): Evaluate h=(b-a)/n

Step(2): for i=0,1,2,…,n

set $x_i = a + ih$ and $y_i = f(x_i)$

Step(3): set s1=0 and s2=0

Step(4): for i=1,2,…,n-1 set

$s1 = s1 + y_i$ (for i is even)

$s2 = s2 + y_i$ (for i is odd)

Step(5): evaluate $I = (h/3).[y_0 + 2.s1 + 4.s2 + y_n]$

Step(6): print I and stop.