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The Distribution of Linear Combinations of normally Distribution

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Univariate Case:

Let $X \sim N(\mu, \sigma^2)$, we want to find the distribution of $Y = cX$, such that c is constant $c > 0$

$\because X \sim N(\mu, \sigma^2)$

$$\therefore f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

To find the distribution of $Y = cX$. We use the transformation method:

$$f(y) = f(x)\Big|_{x = c^{-1}y} |J|$$

whereas: $x = c^{-1}y$

J is Jacobian Coefficient, such that

$|J| = \frac{\partial x}{\partial y}$, therefore $|J|$ means that the absolute value of J

$$-\infty < x < \infty \Rightarrow -\infty < c^{-1}y < \infty$$

$$Y = cX \Rightarrow X = c^{-1}Y \Rightarrow \frac{\partial x}{\partial y} = \frac{1}{c}$$

$$\frac{Y}{c} - \mu \Rightarrow \frac{Y - c\mu}{c}$$

$$|J| = \left|\frac{1}{c}\right| = \frac{1}{c}$$

$$\therefore f(Y) = \frac{1}{(2\pi)\,\sigma}\, e^{-\frac{1}{2}\left(\frac{c^{-1}Y - \mu}{\sigma}\right)^2} \cdot \frac{1}{c}$$

$$\therefore f(Y) = \frac{1}{(2\pi)\,c\,\sigma}\, e^{-\frac{1}{2}\left(\frac{Y - c\mu}{c\,\sigma}\right)^2}$$

This is a p.d.f of normal distribution

$$\therefore Y = cX \sim N(c\mu, c^2\sigma^2)$$

we can check the mean & variance of Y as follows

$$\mu_Y = E(Y) = E(cX) = c\mu$$

$$\sigma_Y^2 = V(Y) = V(cX) = c^2 V(X) = c^2\sigma^2$$

In the multivariate case, we consider the following theorem:

Theorem:

Let X (with p components) be distributed according to $N(\mu, \Sigma)$, then

$$Y = C\underline{X}$$

is distribution according to $N(C\mu, C\Sigma C')$ for C nonsingular