(5) Fixed-point iteration method:

We consider method for determining the solution to an equation f(x)=0 that is expressed for some function g in the form g(x)=x.

A solution to such an equation is said to be a fixed point of the function g. If a fixed point could be found for any given g, then every root-finding problem could also be solved. If we take x_0 as the initial point then the iterative form is

$x_{i+1}=g(x_i)$

that is: $x_1= g(x_0)$, $x_2= g(x_1)$, $x_3= g(x_2)$ and so on.

Example (13): Find the approximate root of the equation x^2-2x-3=0 by using fixed-point method with take x_0=4 .

Solution: We find the exact roots of this equation f(x)= x^2-2x-3=0 $\longrightarrow$

(x-3)(x+1)=0 $\longrightarrow$ x=3 and x=-1

Case(1): f(x)= x^2-2x-3=0 $\longrightarrow$ $x = g_1(x) = \sqrt{3+2x}$

$$\longrightarrow \quad x_{i+1} = g_1(x_i) = \sqrt{3+2x_i}$$

i=0 $\longrightarrow x_1 = g_1(x_0) = \sqrt{3+2x_0} = \sqrt{3+2(4)} = 3.3166$

i=1 $\longrightarrow x_2 = g_1(x_1) = \sqrt{3+2x_1} = \sqrt{3+2(3.3166)} = 3.1037$

i=2 $\longrightarrow x_3 = g_1(x_2) = \sqrt{3+2x_2} = \sqrt{3+2(3.1037)} = 3.0344$

i=3 $\longrightarrow x_4 = g_1(x_3) = \sqrt{3+2x_3} = \sqrt{3+2(3.0344)} = 3.01144$

i=4 $\longrightarrow x_5 = g_1(x_4) = \sqrt{3+2x_4} = \sqrt{3+2(3.01144)} = 3.0038$

$\vdots$

So on , we converge to the root (x=3).

Case(2): x^2-2x-3=0 $\longrightarrow$ x^2-2x=3 $\longrightarrow$ x(x-2)=3 $\longrightarrow$ x=3/(x-2)

$$x = g_2(x) = \frac{3}{x-2}$$

$$\longrightarrow \quad x_{i+1} = g_2(x_i) = \frac{3}{x_i-2}$$

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i=0 ⟶ $x_1 = g_2(x_0) = \frac{3}{x_0-2} = \frac{3}{4-2} = 1.5$

i=1 ⟶ $x_2 = g_2(x_1) = \frac{3}{x_1-2} = \frac{3}{1.5-2} = -6$

i=2 ⟶ $x_3 = g_2(x_2) = \frac{3}{x_2-2} = -0.375$

i=3 ⟶ $x_4 = g_2(x_3) = \frac{3}{x_3-2} = -1.2632$

i=4 ⟶ $x_5 = g_2(x_4) = \frac{3}{x_4-2} = -0.9193$

⋮

So on , we converge oscillating to the root (x=-1).

Case(3): $x^2-2x-3=0$ ⟶ $x^2-3=2x$ ⟶ $x=(x^2-3)/2$

⟶ $x = g_3(x) = \frac{x^2-3}{2}$

⟶ $x_{i+1} = g_3(x_i) = \frac{{x_i}^2-3}{2}$

i=0 ⟶ $x_1 = g_3(x_0) = \frac{{x_0}^2-3}{2} = \frac{(4)^2-3}{2} = 6.5$

i=1 ⟶ $x_2 = g_3(x_1) = \frac{{x_1}^2-3}{2} = 19.625$

i=2 ⟶ $x_3 = g_3(x_2) = \frac{{x_2}^2-3}{2} = 191.0703$

i=3 ⟶ $x_4 = g_3(x_3) = \frac{{x_3}^2-3}{2} = 18252.42977$

⋮

So on , we obtain divergent.

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Note: The following theorem gives sufficient conditions for the existence and uniqueness of a fixed-point.

Theorem:

If $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$, then g has a fixed point in [a,b]. Further, suppose $g'(x)$ exists on [a,b] and $|g'(x)| \leq k < 1$ for all $x \in (a,b)$. Then g has a unique fixed point r in [a,b].

<u>**Example (14):**</u> Find the approximate root of the equation $x^3+4x^2-10=0$ in the interval [1,2] by using fixed-point method with take $x_0=1.5$.

<u>**Solution:**</u> The equation $x^3+4x^2-10=0$ has a unique root in [1,2]. There are many ways to change the equation to the form $x=g(x)$ as follows:

(a) $x = g_1(x) = x - x^3 - 4x^2 + 10$ (b) $x = g_2(x) = \sqrt{\frac{10}{x} - 4x}$

(c) $x = g_3(x) = \frac{1}{2}\sqrt{10 - x^3}$ (d) $x = g_4(x) = \sqrt{\frac{10}{4+x}}$

(e) $x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$ ……………..

To approximate the fixed point of a function g, with the initial approximation $x_0 = 1.5$ and generate the sequence $\{x_i\}_{i=0}^{\infty}$ by letting $x_{i+1} = g(x_i)$ for each $i \geq 0$.

(a) $|g_1'(1.5)| = |-8.75| > 1$ (diverge) (b) $|g_2'(1.5)| = |5.17| > 1$(diverge)

(c) $|g_3'(1.5)| = |0.6556| < 1$(converge) …………………………

If we use $g_3(x)$ to find a fixed point:

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$x_1 = g_3(x_0) = 1.2870$

$x_2 = g_3(x_1) = 1.4025$

$x_3 = g_3(x_2) = 1.3455$

$\vdots$

$x_{20} = g_3(x_{19}) = 1.3652$

Algorithm (Fixed-point method)

Input: x_0, ε, g(x)

Step(1): Set i=1

Step(2): Compute $x = g(x_0)$

Step(3): If $|x - x_0| < \varepsilon$ then print x is a fixed point and stop.

Step(4): Else if set x_0= x and i=i+1 then go to step(2).

<u>Home work:</u>

Find the solution of the equation f(x)= x^2-x-2=0 by using Fixed-point method with x_0=2.5, and $\varepsilon = 5*10^{-5}$.

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