2.4 Simpson's Three-Eighth Method

The derivative of simple Simpson's Three-Eighth rule formula to approximate value of the integral in eq.(1) we approximate the function f(x) as Newton forward interpolation formula and then integration over the interval $[x_0,x_3]$, where n=3 we have

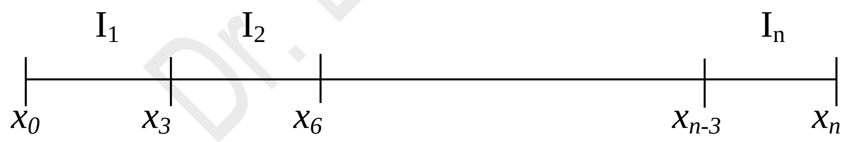
$$\int_{a=x_0}^{b=x_3} f(x)dx = h\int_0^3 f(x_m)dm$$

$$= \frac{3}{8}h[f_0 + 3f_1 + 3f_2 + f_3] \qquad (8)$$

Equation (8) is called **Simple Simpson's Three-Eighth Rule formula** ,the local truncation error in equation (8) is

$$\text{L.T.E.} = \frac{-3h^5}{80} f^4(\theta)\ ,\ \ \theta\epsilon(x_0, x_3)$$

In general, we have subintervals $[x_{i-3}, x_i]$ and we apply eq.(8) at each subinterval we get:



$$\int_{x_i}^{x_{i+3}} f(x)dx \cong\ = \frac{3}{8}h[f_i + 3f_{i+1} + 3f_{i+2} + f_{i+3}]$$

then

$$\int_{a=x_0}^{b=x_n} f(x)dx = \int_{x_0}^{x_3} f(x)dx + \int_{x_3}^{x_6} f(x)dx + \cdots + \int_{x_{n-3}}^{x_n} f(x)dx$$

$$= \frac{3}{8}h[f_0 + 3f_1 + 3f_2 + f_3] + \frac{3}{8}h[f_3 + 3f_4 + 3f_5 + f_6] + \cdots + \frac{3}{8}h[f_{n-3} + 3f_{n-2} + 3f_{n-1} + f_n]$$

$$= \frac{3}{8}h[f_0 + 3\sum_{i=0}^{\frac{n}{3}-1} f(x_{3i+1}) + 3\sum_{i=0}^{\frac{n}{3}-1} f(x_{3i+2}) + 2\sum_{i=1}^{\frac{n}{3}-1} f(x_{3i}) + f_n] \qquad (9)$$

Equation (9) is called **Composite Simpson's Three-Eighth Rule formula**,

and the total truncation error in equation (9) is

$$E_{\frac{3}{8}S} = -\frac{3n}{80}h^5 f^4(\theta) = -\frac{3(b-a)}{80}h^4 f^4(\theta) \quad , \quad \theta \epsilon (a,b)$$

This is **Composite error term** in Composite Simpson's Three-Eighth Rule.

Example(4): Evaluate the following integral using Simpson's Three-Eighth Rule taking $h = \frac{1}{6}$.

$$\int_0^1 \frac{1}{1+x^2}\,dx$$

Solution:

a=0 , b=1 , $f(x) = \frac{1}{1+x^2}$ and $h = \frac{1}{6}$, $n = \frac{b-a}{h} = 6$

$x_i = x_{i-1} + h \Longrightarrow$

$x_0=0$, $f(x_0)=f(0)=1$

$x_1=0+1/6=1/6$, $f(x_1)=f(1/6)=36/37= 0.97297$

$x_2=1/6+1/6=1/3$, $f(x_2)=f(1/3)=9/10=0.9$

$x_3=1/3+1/6=1/2$, $f(x_3)=f(1/2)=4/5=0.8$

$x_4=1/2+1/6=2/3$, $f(x_4)=f(2/3)=9/13=0.6923$

x_5=2/3+1/6=5/6 , $f(x_5)$=f(5/6)=36/61=0.59016

x_6=5/6+1/6=1 , $f(x_4)$=f(1)=1/2=0.5

$$I[f]=\int_0^1 \frac{1}{1+x^2}dx=\frac{3}{8}h[f_0+3\sum_{i=0}^{\frac{n}{3}-1} f(x_{3i+1})+3\sum_{i=0}^{\frac{n}{3}-1} f(x_{3i+2})+2\sum_{i=1}^{\frac{n}{3}-1} f(x_{3i})+f_n]$$

$$=\frac{3h}{8}[1+3f_1+3f_2+2f_3+3f_4+3f_5+f_6]$$

$$=\frac{3h}{8}\ [1+3(\ f_1+f_2+f_4+f_5)+2f_3+f_6]$$

$$=0.7854$$

Example(5): Use Simpson's Three-Eighth Rule to evaluate the integral

$$\int_0^1 (x^3+1)dx$$

Considering h=1/6.

Solution: H.W.

Algorithm of Simpson's Three-Eighth rule:

Input: a , b , n , f(x)

Step(1): evaluate h=(b-a)/n

Step(2): for i=0,1,2,…,n

set x_i=a +ih

Step(3): evaluate the value of the integral

I= $\frac{3}{8}h[f_0+3\sum_{i=0}^{\frac{n}{3}-1} f(x_{3i+1})+3\sum_{i=0}^{\frac{n}{3}-1} f(x_{3i+2})+2\sum_{i=1}^{\frac{n}{3}-1} f(x_{3i})+f_n]$

Step(4): print I and stop.

2.5 Midpoint Method

In this method one point is used, in particular

$$x_0 = \frac{a+b}{2} \ , \quad p_0(x) = f(\frac{a+b}{2})$$

$$\int_a^b f(x)dx = \int_a^b p_0(x)dx = (b-a)\, f(\frac{a+b}{2}) \qquad (10)$$

The error term corresponding to midpoint rule is

$$E \quad = (b-a)^3(\frac{f''(\theta)}{24})$$

In general, we apply eq.(10) at each subintervals $[x_{i-1}, x_i]$ and we get

$$\int_a^b f(x)dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\, dx = h \sum_{i=1}^{n} f(\frac{x_{i-1}+x_i}{2}) \qquad (11)$$

Where h=(b-a)/n and the composite error of equation (11) is

$$E_M = (b-a)(\frac{h^2}{24})f''(\theta)$$

Example(6): Use Midpoint method to evaluate the integral

$$\int_0^2 e^x dx$$

Considering h=0.25.

Solution:

Clearly, a=0 , b=2 , $f(x) = e^x$ and $h = 0.25$

, $n = \frac{b-a}{h} = 8$

$x_i=x_{i-1}+h$ ⟹

$x_0=0$, $x_3=0.5+0.25=0.75$, $x_6=1.25+0.25=1.5$

$x_1=0+0.25=0.25$, $x_4=0.75+0.25=1$, $x_7=1.5+0.25=1.75$

$x_2=0.25+0.25=0.5$, $x_5=1+0.25=1.25$, $x_8=1.75+0.25=2$

$$\int_0^2 e^x dx = h\sum_{i=1}^{8} e^{\frac{x_{i-1}+x_i}{2}}$$

$$= h(e^{\frac{x_0+x_1}{2}} + e^{\frac{x_1+x_2}{2}} + e^{\frac{x_2+x_3}{2}} + e^{\frac{x_3+x_4}{2}} + e^{\frac{x_4+x_5}{2}} + e^{\frac{x_5+x_6}{2}} + e^{\frac{x_6+x_7}{2}} + e^{\frac{x_7+x_8}{2}})$$

$$= 0.25(e^{0.125} + e^{0.375} + e^{0.625} + e^{0.875} + e^{1.125} + e^{1.375} + e^{1.625} + e^{1.875})$$

=0.25(1.1331+1.4550+1.8682+2.3989+3.0802+3.9551+5.0784+6.5208)

=0.25(25.4897)

=6.3724

Example(7): Evaluate the integration $\int_0^1 (x^3+1)dx$ using Midpoint method with h=0.25.

Solution: H.W.

Midpoint Algorithm:

Input: a , b , n , f(x)

Step(1): evaluate h=(b-a)/n

Step(2): for i=0,1,2,…,n

set x_i=a+ih

Step(3): set sum=0

Step(4): for i=1,2,…,n set

sum =sum + $f(\frac{x_{i-1}+x_i}{2})$

Step(5): evaluate I=h*sum

Step(6): print I and stop.