$$\frac{Y}{c} - \mu \Rightarrow \frac{Y - c\mu}{c}$$

$$|J| = \left|\frac{1}{c}\right| = \frac{1}{c}$$

$$\therefore f(Y) = \frac{1}{(2\pi)\sigma} e^{-\frac{1}{2}\left(\frac{c^{-1}Y - \mu}{\sigma}\right)^2} \cdot \frac{1}{c}$$

$$\therefore f(Y) = \frac{1}{(2\pi) c\sigma} e^{-\frac{1}{2}\left(\frac{Y - c\mu}{c\sigma}\right)^2}$$

This is a p.d.f of normal distribution

$$\therefore Y = cX \sim N(c\mu, c^2\sigma^2)$$

we can check the mean & variance of Y as follows

$$\mu_Y = E(Y) = E(cX) = c\mu$$

$$\sigma_Y^2 = V(Y) = V(cX) = c^2 V(X) = c^2\sigma^2$$

In the multivariate case, we consider the following theorem:

Theorem:
Let X (with p components) be distributed according to $N(\mu, \Sigma)$, then

$$\underline{Y} = C\underline{X}$$

is distribution according to $N(C\mu, C\Sigma C')$ for C nonsingular

Proof

$$\underline{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{pmatrix}, \quad \underset{p\times 1}{\underline{Y}} = \underset{p\times p}{C} \; \underset{p\times 1}{\underline{X}}$$

$$\underline{Y} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & & & \\ c_{p1} & c_{p2} & \cdots & c_{pp} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{p} c_{1i} X_i \\ \sum_{i=1}^{p} c_{2i} X_i \\ \vdots \\ \sum_{i=1}^{p} c_{pi} X_i \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_p \end{pmatrix}$$

$\underline{X} \sim N(\underline{\mu}, \Sigma)$

$$f(\underline{X}) = \frac{1}{(2\pi)^{p/2} \, |\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}(\underline{X}-\underline{\mu})' \Sigma^{-1} (\underline{X}-\underline{\mu})} \quad \text{--- (1)}$$

$$f(\underline{Y}) = f(X)\Big|_{X = C^{-1}Y} |J| \quad \text{--- (2)}$$

$|J|$ is the absolute value of J

$$|J| = \begin{vmatrix} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} & \cdots & \frac{\partial X_1}{\partial Y_p} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} & \cdots & \frac{\partial X_2}{\partial Y_p} \\ \vdots & & & \\ \frac{\partial X_p}{\partial Y_1} & \frac{\partial X_p}{\partial Y_2} & \cdots & \frac{\partial X_p}{\partial Y_p} \end{vmatrix}$$

$$\underline{Y} = C\underline{X} \Rightarrow \underline{X} = C^{-1}\underline{Y} \Rightarrow \frac{\partial \underline{X}}{\partial \underline{Y}} = C^{-1}$$

$$\Rightarrow |J| = \left|\frac{\partial \underline{X}}{\partial \underline{Y}}\right| = |C^{-1}| = \frac{1}{|C|} = \sqrt{\frac{1}{|C|^2}}$$

$$= \sqrt{\frac{1}{|C||C'|}} = \sqrt{\frac{|\Sigma|}{|C||\Sigma||C'|}}$$

$$|J| = \frac{|\Sigma|^{\frac{1}{2}}}{|C\Sigma C'|^{\frac{1}{2}}}$$

substitute in ② we get:

$$f(Y) = \frac{1}{(2\pi)^{P/2}|\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}(C^{-1}Y-\mu)'\Sigma^{-1}(C^{-1}Y-\mu)}$$

$$* \frac{|\Sigma|^{\frac{1}{2}}}{|C\Sigma C'|^{\frac{1}{2}}}$$

Now:

$$(C^{-1}Y-\mu)'\Sigma^{-1}(C^{-1}Y-\mu)$$

$$\Rightarrow (C^{-1}Y - C^{-1}C\mu)\ \Sigma^{-1}(C^{-1}Y - C^{-1}C\mu) \qquad C^{-1}C = I$$

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$$\Rightarrow [C^{-1}(Y-C\mu)]'\ \Sigma^{-1}\ [C^{-1}(Y-C\mu)]$$

$$(Y-C\mu)'(C^{-1})'\ \Sigma^{-1}\ C^{-1}(Y-C\mu)$$

we used the property of the transpose :

$(ABC)' = C'B'A'$

Now: we used the properties $(C^{-1})' = (C')^{-1}$

and $(C')^{-1} \Sigma^{-1} C^{-1} = (C \Sigma C')^{-1}$, we get :

$$f(\underline{Y}) = \frac{1}{(2\pi)^{p/2} |C \Sigma C'|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2} (Y - C\mu)' (C \Sigma C')^{-1} (Y - C\mu)\right. \quad \text{--- (3)}$$

Eq (3) represents the p.d.f of multivariate normal dist^n. whose mean is $C\underline{\mu}$ and var-cov. matrix $C \Sigma C'$ i.e: $Y \sim N_p(C\underline{\mu}, C\Sigma C')$

<u>Remarks</u> :

If $C = I_p$ Then $f(\underline{Y}) = f(\underline{X})$