

2.6 Romberg Method

The Romberg method generally generates a lower trigonometric matrix of numbers, all of which are approximations of the value of integration

$$\begin{array}{ccccccc}
 R(1,1) & & & & & & \\
 R(2,1) & R(2,2) & & & & & \\
 R(3,1) & R(3,2) & R(3,3) & & & & \\
 R(4,1) & R(4,2) & R(4,3) & R(4,4) & & & \\
 \cdot & \cdot & \cdot & \cdot & & & \\
 R(n,1) & R(n,2) & R(n,3) & R(n,4) & \dots & R(n,n) &
 \end{array}$$

The numbers in the first column of the matrix represent the approximate initial values of integration after applying the trapezoidal rule i.e.:

$$R(1,1) = \frac{b-a}{2} [f(a) + f(b)]$$

The rest of the elements of the first column are calculated from the relationship:

$$R(k, 1) = \frac{1}{2} \left[R(k-1, 1) + h_{k-1} \sum_{i=1}^{2^{k-2}} f\left(a + \left(i - \frac{1}{2}\right)h_{k-1}\right) \right], \quad k = 2, 3, \dots, n$$

$$h_{k-1} = \frac{b-a}{2^{k-2}}$$

The rest of the elements are calculated from the law:

$$R(k, i) = \frac{4^{i-1} * R(k, i-1) - R(k-1, i-1)}{4^{i-1} - 1}, \quad k, i = 2, 3, \dots, n$$

Example(8): Find the approximation value of the integral by using Romberg integration method

$$\int_0^2 x^2 dx, \quad n=4$$

Solution: Clearly, $a=0$, $b=2$, $f(x) = x^2$

$$R(1,1) = \frac{b-a}{2} [f(a) + f(b)] = \frac{2-0}{2} [4 + 0] = 4$$

$$R(k, 1) = \frac{1}{2} \left[R(k-1, 1) + h_{k-1} \sum_{i=1}^{2^{k-2}} f\left(a + \left(i - \frac{1}{2}\right)h_{k-1}\right) \right], \quad k = 2, 3, 4$$

$$h_{k-1} = \frac{b-a}{2^{k-2}}$$

$$\text{Let } k=2 \text{ then } h_1 = \frac{2-0}{2^{2-2}} = 2$$

$$\begin{aligned} R(2,1) &= \frac{1}{2} \left[R(1,1) + h_1 \sum_{i=1}^{2^{2-2}} f\left(0 + \left(i - \frac{1}{2}\right)h_1\right) \right] \\ &= \frac{1}{2} \left[4 + 2 * f\left(0 + \left(0 + \frac{1}{2} * 2\right)\right) \right] = \frac{6}{2} = 3 \end{aligned}$$

$$k=3 \text{ then } h_2 = \frac{2-0}{2^{3-2}} = 1$$

$$\begin{aligned} R(3,1) &= \frac{1}{2} \left[R(2,1) + h_2 \sum_{i=1}^{2^{3-2}} f\left(0 + \left(i - \frac{1}{2}\right)h_2\right) \right] \\ &= \frac{1}{2} \left[3 + 1 * \{f\left(0 + \frac{1}{2} * 1\right) + f\left(0 + \frac{3}{2} * 1\right)\} \right] \\ &= \frac{1}{2} \left[3 + \frac{1}{4} + \frac{9}{4} \right] = \frac{11}{4} = 2.75 \end{aligned}$$

$$k=4 \text{ then } h_3 = \frac{2-0}{2^{4-2}} = \frac{1}{2}$$

$$\begin{aligned} R(4,1) &= \frac{1}{2} \left[R(3,1) + h_3 \sum_{i=1}^{2^{4-2}} f\left(0 + \left(i - \frac{1}{2}\right)h_3\right) \right] \\ &= \frac{1}{2} \left[2.75 + \frac{1}{2} * \left\{ f\left(0 + \frac{1}{2} * \frac{1}{2}\right) + f\left(0 + \frac{3}{2} * \frac{1}{2}\right) + f\left(0 + \frac{5}{2} * \frac{1}{2}\right) + \right. \right. \\ &\quad \left. \left. f\left(0 + \frac{7}{2} * \frac{1}{2}\right) \right\} \right] \\ &= 2.6873 \end{aligned}$$

Now, we evaluate the values

$$R(k,i) = \frac{4^{i-1} * R(k,i-1) - R(k-1,i-1)}{4^{i-1} - 1}, \quad k,i = 2,3,4$$

$$\begin{aligned} R(2,2) &= \frac{4^{2-1} * R(2,1) - R(1,1)}{4^{2-1} - 1} = \frac{4 * R(2,1) - R(1,1)}{4^{2-1} - 1} \\ &= \frac{4*3-4}{4-1} = \frac{8}{3} = 2.6667 \end{aligned}$$

$$R(3,2) = \frac{4^{2-1} * R(3,1) - R(2,1)}{4^{2-1} - 1} = \frac{8}{3} = 2.6667$$

And so on.

Example(9): Find the approximation value of the integral by using Romberg integration method

$$\int_0^{\frac{\pi}{4}} \sec x \, dx, \quad n=5$$

Solution: H.W.

Romberg Integration Algorithm

Input: a , b , n , f(x)

Step(1): evaluate $h=b-a$

Step(2): find $R(1,1) = \frac{b-a}{2} [f(a) + f(b)]$

Step(3): for $k=2,3,4,\dots,n$ find $h_{k-1} = \frac{b-a}{2^{k-2}}$ and

$$R(k, 1) = \frac{1}{2} \left[R(k-1, 1) + h_{k-1} \sum_{i=1}^{2^{k-2}} f(a + (i - \frac{1}{2})h_{k-1}) \right]$$

Step(4): for $i=2,3,\dots,n$ and $k=i,i+1,\dots,n$ evaluate

$$R(k, i) = \frac{4^{i-1} * R(k, i-1) - R(k-1, i-1)}{4^{i-1} - 1}$$

Step(5): print $R(n,n)$ and stop.
