Remark:

If $C = I_p$ Then $\mathcal{L}(\underline{Y}) = \mathcal{L}(\underline{X})$

Ex Let $\underline{X} \sim N_2(\mu, \Sigma)$ where $\mu = (2 \quad 3)'$

and $\Sigma = \begin{pmatrix} 1 & 1 \\ & 2 \end{pmatrix}$. Find the Probability

distribution of the random vector $\underline{Y} = (Y_1 \quad Y_2)'$, where

$$Y_1 = X_1 + 2X_2$$
$$Y_2 = X_1 + 3X_2$$

Sol

$$\underline{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = C\underline{X}$$

$$\underline{Y} \sim N_2(C\mu, C\Sigma C')$$

$$E\underline{Y} = C\mu = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \end{pmatrix}$$

$$\text{var } \underline{Y} = C\Sigma C' = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 18 \\ & 25 \end{pmatrix}$$

Lemma (1)

If $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$, then the Pobability dist of (q) Linear Combination of the variates of a r.v $\underline{X}$ where $q < P$ is a multivariate normal dist with q variates.

For example $\underline{Y} = A\underline{X}$ where A is a matrix of degree $(q \times p)$ and $\underline{Y} = (Y_1 \; Y_2 \; -- \; Y_q)$ i.e

$$\underline{Y} = A\underline{X} = \begin{pmatrix} a_{11} & a_{12} & --- & a_{1p} \\ a_{21} & a_{22} & --- & a_{2p} \\ \vdots & & & \\ a_{q1} & --- & & a_{qp} \end{pmatrix}_{q \times p} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{pmatrix}_{p \times 1}$$

$$\sim N_q(A\underline{\mu}, A \Sigma A')$$

and if $\underline{Y} = \underline{X} + \underline{d}$ where $\underline{d}$ is a vector of Constants of degree $(P \times 1)$ then $\underline{Y} \sim N_p(\underbrace{\underline{\mu} + \underline{d}}_{E(\underline{Y})}, \underbrace{\Sigma}_{V(\underline{Y})})$

Ex.

If $\underline{X} \sim N_3(\underline{\mu}, \Sigma)$ where $\underline{\mu} = (1 \; 0 \; 1)$ and

$$\Sigma = \begin{pmatrix} 2 & 1 & 0 \\ & 3 & 1 \\ & & 2 \end{pmatrix}_{3 \times 3}$$

Find the dist of the following

1- $\underline{Y} = \begin{pmatrix} X_1 + X_2 \\ X_2 - X_3 \end{pmatrix}$

(2) $W = \underline{Y} + \underline{d}$ where $\underline{d} = (1 \; 0)'$

Solu.

$$\underline{Y} = A\underline{X} = \underset{\substack{2\times3\\ q\times p}}{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}} \underset{\substack{3\times1\\ p\times1}}{\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}} \sim N_2 \sim (A\mu, A\Sigma A'$$

$$E(\underline{Y}) = A\mu = \underset{2\times3}{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}} \underset{3\times1}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}} = \underset{2\times1}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} = \begin{bmatrix} EY_1 \\ EY_2 \end{bmatrix}$$

$$var(\underline{Y}) = A\Sigma A'$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ 3 & 3 \end{pmatrix}$$

(2) $\underline{W} = \underline{Y} + \underline{d} \sim N_2(E(\underline{Y}) + \underline{d},\ var(\underline{Y}))$

$$E(\underline{W}) = E(\underline{Y}) + \underline{d} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} EW_1 \\ EW_2 \end{pmatrix}$$

$$v(\underline{W}) = var(\underline{Y}) = \begin{pmatrix} 7 & 3 \\ 3 & 3 \end{pmatrix}$$

Lemma (2)

If $\underline{X} \sim N_p(\mu, \Sigma)$ then the p.d.f of linear combination $a_1 X_1 + a_2 X_2 + \cdots + a_p X_p = a^* \underline{X}$ where $a^* = (a_1 \; a_2 \cdots a_p)'$ is a row vector distributed a univariate normal dist with mean $\underline{a}^* \underline{\mu}$ and variance $a^* \Sigma a^{*\prime}$, i.e:-

$\underline{a}^* \underline{X} \sim N(a^* \mu, a^* \Sigma a^{*\prime})$

For example,

$\underline{a}^* = (1 \; 0 \; 0 \cdots 0)$ is a row vector of degree $(1 \times p)$ then $\underline{a}^* \underline{X} = X_1 \sim N(\mu_1, \sigma_1^2)$

If $a^* = (0 \; 1 \; 0 \; 0 \cdots 0) \Rightarrow \underline{a}^* \underline{X} = X_2 \sim N(\mu_2, \sigma_2^2)$

Therefore if $\underline{a}^*$ has only one element $a_j = 1$ and another element, $a_1 = a_2 = \cdots = a_{j-1} = a_{j+1} = \cdots = a_p = 0$

Then the linear combination $\underline{a}^* \underline{X} = X_j \sim N(\mu_j, \sigma_j^2)$

ex If $\underline{X} \sim N_3(\mu, \Sigma)$ Find the distributions of the following :-

① $W = X_1 + X_2$ $\leftarrow$ عبارة عن بيانات واحده

② $Y = (X_1 + X_2 \quad X_1 - X_2)'$

Solu

① $W = \underline{a}^{*} \underline{X} = (1 \;\; 1 \;\; 0) \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N(a^{*}\mu, \underline{a}^{*} \Sigma \underline{a}$

where $E(\underline{W}) = a^{*}\mu = (1 \;\; 1 \;\; 0) \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \mu_1 + \mu$

and $v(\underline{W}) = \underline{a}^{*} \Sigma \underline{a}^{*'}$

$$= (1 \;\; 1 \;\; 0) \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ & \sigma_2^2 & \sigma_{23} \\ & & \sigma_3^2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \sigma_1^2 + \sigma_2^2 + 2\sigma_{12}$$

② $\underline{Y} \sim N_2(A\mu, A \Sigma A')$

where $\underline{Y} = A\underline{X} = \underset{2\times 3}{\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$

$$E(\underline{Y}) = A\underline{\mu} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \underset{3\times}{\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}} = \begin{pmatrix} \mu_1 + \mu_2 \\ \mu_1 - \mu_2 \end{pmatrix}$$

$$A \Sigma A' = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ & \sigma_2^2 & \sigma_{23} \\ & & \sigma_3^2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 + \sigma_2^2 + 2\sigma_{12} & \sigma_1^2 - \sigma \\ & \sigma_1^2 + \sigma_2^2 - 2\sigma \end{pmatrix}$$