Maximum Likelihood method:

اذا توفرت لدينا مشاهدات عينة عشوائية ذات حجم (n) من متغيرات الاستجابة والتوضيحية . اذا ان:

$$\underline{Y}_i = B X_i + u_i \quad for \quad i = 1, 2, \ldots, n$$

where

$\underline{u}_i \sim N_p(0, \Sigma)$ then $\boxed{Y_i \sim N_p(B \underline{x}_i, \Sigma)}$

for $i = 1, 2, \ldots, n$

Then the p.d.f of $(\underline{Y}_i)$ is

$$f(Y_i) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}(\underline{Y}_i - \underline{B} X_i)' \Sigma^{-1} (Y_i - B \underline{X}_i)}$$

$$\therefore L(\cdot, \cdot) = (2\pi)^{-\frac{np}{2}} |\Sigma|^{-\frac{n}{2}} e^{-\frac{1}{2}\sum(\underline{Y}_i - \underline{B}\underline{X}_i)' \Sigma^{-1} (\underline{Y}_i - B \underline{X}_i)}$$

we will study three cases:

Case (1): when Σ known

$$L(B) = (2\pi)^{-\frac{np}{2}} |\Sigma|^{-\frac{n}{2}} e^{-\frac{1}{2}\sum_{i=1}^{n}(\underline{Y}_i - B \underline{X}_i)' \Sigma^{-1} (\underline{Y}_i - B \underline{X}_i)}$$

$$\ln L(B) = \frac{-np}{2} \ln(2\pi) - \frac{n}{2} \ln|\Sigma| - \frac{1}{2} \sum_{i=1}^{n} (\underline{Y}_i - B \underline{X})' \Sigma^{-1} (\underline{Y}_i - B \underline{X}_i)$$

Remarke:

if you have ~~the~~ a quadratic form

$Q = (\underline{Z} - A \underline{c})' w (\underline{Z} - A \underline{c})$ when w is a symetric and Z, c be vector of the same size then

$$\frac{\partial Q}{\partial A} = -2W(\underline{Z} - A\underline{C})\underline{C}' \quad --- \quad (a)$$

باستخدام الفائدة (a) في مستقة لوغاريتم دالة الإمكانة نسبة الى المصفوفة B نحصل على:

$$\frac{\partial \ln L(B)}{\partial B} = (2)\left(\tfrac{1}{2}\right) \sum_{i=1}^{n} \Sigma^{-1} (Y_i - BX_i)X_i'$$

$$\frac{\partial \ln L(B)}{\partial B} = 0 \Longrightarrow \Sigma^{-1} \sum_{i=1}^{n} (\underline{Y}_i - BX_i)\underline{X}_i' = 0$$

نضرب طرفي المعادلة بـ (Σ) لأن (Σ) غير شاذة

$$\sum_{i=1}^{n} (Y_i - BX_i)X' = 0$$

$$\sum_{i=1}^{n} Y_i X' = +B \sum_{i=1}^{n} X_i X_i' \quad ------ \quad (*)$$

But $\sum X_i X'$ is a full rank matrix this mean it has inverse which is $(\sum X_i X_i')^{-1}$

Multiplying both sides of $(*)$ from the right by $(\sum X_i X_i')^{-1}$ we get

$$\hat{B}_{mle} = \sum Y_i X_i' \left(\sum (X_i X')\right)^{-1} \Rightarrow \hat{B}_{m.l.e} = YX'(XX')^{-1}$$

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Case (2) when B is know

$$L(\Sigma) = (2\pi)^{-\frac{np}{2}} |\Sigma|^{-\frac{n}{2}} e^{-\frac{1}{2}\sum (\underline{Y}_i - B\underline{X}_i)' \Sigma^{-1} (\underline{Y}_i - B\underline{X}_i)}$$

$$= (2\pi)^{-\frac{np}{2}} |\Sigma|^{-\frac{n}{2}} e^{-\frac{1}{2} tr\, \Sigma^{-1} \sum_{i=1}^{n} (\underline{Y}_i - B\underline{X}_i)(\underline{Y}_i - B\underline{X}_i)'}$$

$$\ln L(\Sigma) = -\frac{np}{2}(2\pi) - \frac{n}{2}|\Sigma| - \frac{1}{2} tr\, \Sigma^{-1} \sum_{i=1}^{n} (\underline{Y}_i - B\underline{X}_i)(\underline{X}_i - B\underline{X}_i)'$$

$$\frac{\partial \ln L(\Sigma)}{\partial \Sigma} = -\frac{n}{2}\Sigma + \frac{1}{2}\Sigma^{-1} \sum_{i=1}^{n} (Y_i - BX_i)(X_i - BX_i)' \Sigma^{-1}$$

$$\frac{\partial \ln L(\Sigma)}{\partial \Sigma} = 0 \Rightarrow \hat{\Sigma} = \frac{\sum_{i=1}^{n} (Y_i - BX_i)(Y_i - BX_i)'}{n}$$

$$\Rightarrow \boxed{\hat{\Sigma} = \frac{(Y - BX)(Y - BX)'}{n}}$$

$\hat{\Sigma}$ is unbiase estimation for Σ

$$E\hat{\Sigma} = \frac{1}{n} \sum_{i=1}^{n} \underbrace{E(\underline{Y}_i - B\underline{X}_i)(\underline{Y}_i - B\underline{X}_i)'}_{=\Sigma}$$

$$= \frac{1}{n} \sum_{i=1}^{n} \Sigma = \frac{n}{n}\Sigma$$

$$\therefore E\hat{\Sigma} = \Sigma$$

$\therefore \hat{\Sigma}$ is unbiase estimation

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Case (3) when Σ, B unknown

$$L(B,\Sigma) = (2\pi)^{-\frac{np}{2}} |\Sigma|^{-\frac{n}{2}} e^{-\frac{1}{2}\sum_{i=1}^{n}(Y_i - BX_i)'\Sigma^{-1}(Y_i - BX_i)}$$

$$\ln L(B,\Sigma) = -\frac{np}{2}\ln 2\pi - \frac{n}{2}|\Sigma| - \frac{1}{2}\sum_{i=1}^{n}(Y_i - BX_i)'\Sigma^{-1}(Y_i - BX_i)$$

$$\frac{\partial \ln L(B,\Sigma)}{\partial B} = -\frac{1}{2}\sum_{i=1}^{n}(\underline{Y}_i - B\underline{X}_i)'\Sigma^{-1}(\underline{Y}_i - B\underline{X}_i)$$

$$\frac{\partial \ln L(B,\Sigma)}{\partial \Sigma} = \frac{-n}{2}\frac{\partial}{\partial \Sigma}(\ln \Sigma) - \frac{1}{2}\frac{\partial}{\partial \Sigma}\, tr\, \Sigma \sum_{i=1}^{n}(\underline{Y}_i - B\underline{X})(\underline{Y} - B\underline{X})'$$

$$= \frac{n}{2}\Sigma^{-1} + \frac{1}{2}\Sigma^{-1}\sum_{i=1}^{n}(\underline{Y}_i - B\underline{X}_i)(\underline{Y}_i - B\underline{X}_i)'$$

$$\frac{\partial \ln L(B,\Sigma)}{\partial B} = 0, \quad \frac{\partial \ln L(B,\Sigma)}{\partial \Sigma}$$

بعد حل المعادلتين نحصل على تقديرات لـ B ، Σ

(البرهان غير مطلوب)

$$\therefore \hat{B} = YX'(XX')^{-1}$$

$$\hat{\Sigma} = \frac{(Y - \hat{B}X)(Y - \hat{B}X)'}{n}$$

In this case $\hat{\Sigma}$ is abiased estimator

$$E\hat{\Sigma} = \frac{1}{n} E(Y - \hat{B}X)(Y - \hat{B}X)' \Rightarrow (n-1)\Sigma$$

$$\therefore E\hat{\Sigma} = \frac{n-1}{n}\Sigma \quad \therefore \hat{\Sigma} \text{ is abiased esti.}$$