Chapter Three
Numerical Solution of Linear Systems

There are two methods for solving linear system of equations:

(i) **Direct method**
- Gauss eliminations
- Gauss elimination with partial pivoting
- Gauss-Jordan elimination
- Cramer Rule
- Decomposition method

(ii) **Indirect method (iterative methods)**
- Jacobi method
- Gauss-Seidel method

(i) Direct method:

(1) Gauss elimination with partial pivoting

Algorithm for Gauss elimination: To solve n×n system

R1: $a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n=a_{1,n+1}$
R2: $a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n=a_{2,n+1}$
$\vdots$
Rn: $a_{n1}x_1+a_{n2}x_2+\ldots a_{nn}x_n=a_{n,n+1}$
Augmented matrix $A=(a_{ij})$, where $1\le i\le n$, $1\le j\le n+1$.

Step 1: For i=1, 2, …,n-1 do step 2-4 (Elimination Process)
Step 2: Let p be the smallest integer with $1\le p\le n$ and $a_{pi}\neq 0$.
If no integer p can be found
Then output ('no unique solution exists');
Stop
Step 3: If $p\neq I$ then perform $(R_p)\leftrightarrow(R_i)$.
Step 4: For j=i+1, i+2, …,n do step 5 and 6.
Step 5: Set $m_{ji}=\dfrac{a_{ji}}{a_{ii}}$.
Step 6: perform $(R_j-m_{ji}R_i)\rightarrow R_j$.
Step 7: If $a_{nn}=0$ then output ('no unique solution exists');
Stop

Step 8: set $x_n = \frac{a_{n,n+1}}{a_{nn}}$ (Start backward substitution)

Step 9: For i=n-1, n-2, …,1 set $x_i = \frac{a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}}$.

Step 10: Output (x_1, x_2, …,x_n); (procedure complete successfully)
Stop.

Example 1:

Use Gauss elimination to solve

$2x_1 - x_2 + x_3 = -1$

$3x_1 + 3x_2 + 9x_3 = 0$

$3x_1 + 3x_2 + 5x_3 = 4.$

Solution:

$$\begin{matrix} R1: \\ R2: \\ R3: \end{matrix}\left(\begin{array}{ccc|c} 2 & -1 & 1 & -1 \\ 3 & 3 & 9 & 0 \\ 3 & 3 & 5 & 4 \end{array}\right)$$

$a_{11}=2$, $m_{ji} = \frac{a_{ji}}{a_{ii}} \Rightarrow m_{21} = \frac{a_{21}}{a_{11}} = \frac{3}{2}$; $m_{31} = \frac{a_{31}}{a_{11}} = \frac{3}{2}$

$(R_2 - m_{21}R_1) \to R_2$

$(R_3 - m_{31}R_1) \to R_3$

$$\begin{matrix} R1: \\ R2: \\ R3: \end{matrix}\left(\begin{array}{ccc|c} 2 & -1 & 1 & -1 \\ 0 & 9/2 & 15/2 & 3/1 \\ 0 & 9/2 & 7/2 & 11/2 \end{array}\right),$$

$a_{22} = \frac{9}{2}$, $m_{ji} = \frac{a_{ji}}{a_{ii}} \Rightarrow m_{32} = \frac{a_{32}}{a_{33}} = \frac{9/2}{9/2} = 1$

$(R_3 - m_{32}R_2) \to R_3$

$$\begin{matrix} R1: \\ R2: \\ R3: \end{matrix}\left(\begin{array}{ccc|c} 2 & -1 & 1 & -1 \\ 0 & 9/2 & 15/2 & 3/1 \\ 0 & 0 & -4 & 4 \end{array}\right)$$

$x_3 = \frac{a_{33}}{a_{34}} = \frac{-4}{4} = -1$, $x_2 = \frac{\frac{3}{2} - \frac{15}{2}x_3}{\frac{9}{2}} = 2$, $x_1 = \frac{-1 + x_2 - x_3}{2} = 1.$

Example 2: Solve

$4x_1+\ x_2+x_3=6$
$2x_1+5x_2+2x_3=3$
$x_1+2x_2+4x_3=11$

The exact solution to each system is $x_1=1$, $x_2=-1$, $x_3=3$.

Gauss elimination with partial pivoting:

Example 3:

The linear system

R1: $0.003x_1+59.14x_2=59.17$
R2: $5.291x_1-6.130x_2=46.78$

has the exact solution $x_1=10$ and $x_2=1$.

Solution: (by Gauss elimination) ; $m_{21}=\frac{a_{21}}{a_{11}}=\frac{5.291}{0.003}=1763.66$

$(R_2-m_{21}R_1)\to R_2$ R1: $0.003x_1+59.14x_2=59.17$
R2: $-104300x_2=-104400$

Backward substitution implies $x_2=1.001$ and $x_1=-10$.
We note that the large error in the numerical solution for x_1 resulted from the small error of 0.001 in solving for x_2. Above example illustrates that difficulties can arise in some case when the pivot element $a^{(k)}_{k,k}$ is small relative to entries $a^{(k)}_{i,j}$, for $k\le i\le n$ and $k\le j\le n$. Partial pivoting in general are accomplished by selecting a new element for the pivot $a^{(k)}_{pq}$ and interchanging the k^{th} and p^{th} rows, followed by the interchange of the k^{th} and q^{th} columns, if necessary. The simplest strategy is to select the element in the same column that is below the diagonal and has the largest absolute value that is determine p such that $\left|a^{(k)}_{pk}\right|=\max_{k\le i\le n}\left|a^{(k)}_{ik}\right|$ and perform $(Rk)\leftrightarrow(Rp)$.

Example 4:

Reconsider the linear system
R1: $0.003x_1+59.14x_2=59.17$
R2: $5.291x_1-6.130x_2=46.78$

Solution: Use the pivoting procedure:

$$\max\left\{\left|a^{(1)}_{11}\right|,\left|a^{(1)}_{21}\right|\right\}=\max\{0.003,5.291\}=5.291.$$

The operation (R2)↔(R1) is performed to give the system

$5.291x_1 - 6.130x_2 = 46.78$

$0.003x_1 + 59.14x_2 = 59.17$; $\quad m_{21} = \dfrac{a_{21}^{(1)}}{a_{11}^{(1)}} = \dfrac{0.003}{5.291} = 0.0005670$

The operation (R2-m21R1)→R1 reduce the system to

$5.291x_1 - 6.130x_2 = 46.78$

$\qquad\qquad 59.14x_2 = 59.14$

$\Rightarrow x_2 = 1$ and $x_1 = 1$

(2) Gauss-Jordan elimination method

Example 5:

Use Gauss-Jordan elimination method to solve

$$\begin{aligned} 4x_1 - 9x_2 + 2x_3 &= 5 \\ 2x_1 - 4x_2 + 6x_3 &= 3 \\ x_1 - x_2 + 3x_3 &= 4. \end{aligned}$$

Solution:

$$\begin{matrix} R1: \\ R2: \\ R3: \end{matrix} \left(\begin{array}{ccc|c} 4 & -9 & 2 & 5 \\ 2 & -4 & 6 & 3 \\ 1 & -1 & 3 & 4 \end{array}\right)$$

$$\begin{matrix} (R_2 - m_{21}R_1) \to R_2 \\ (R_3 - m_{31}R_1) \to R_3 \\ (R_3 - m_{32}R_2) \to R_3 \end{matrix} \quad \begin{matrix} R1: \\ R2: \\ R3: \end{matrix} \left(\begin{array}{ccc|c} 4 & -9 & 2 & 5 \\ 0 & 0.5 & 5 & 0.5 \\ 0 & 0 & -10 & 1.5 \end{array}\right), \quad R1/10$$

$$\begin{matrix} R1: \\ R2: \\ R3: \end{matrix} \left(\begin{array}{ccc|c} 4 & -9 & 0 & 5.3 \\ 0 & 1 & 0 & 2.5 \\ 0 & 0 & 1 & -0.15 \end{array}\right), \quad R2/0.5$$

$$\begin{matrix} R1: \\ R2: \\ R3: \end{matrix} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 6.95 \\ 0 & 1 & 0 & 2.5 \\ 0 & 0 & 1 & -0.15 \end{array}\right), \quad R1/4$$

$$x_1 = 6.95, \; x_2 = 2.5, \; x_3 = -0.15$$