2.7 Gauss-Legendre Integration

If f(x) is continuous on the interval [-1,1] then Gauss-Legendre two-point rule is

$$\int_{-1}^{1} f(x)\,dx \cong G_2(f) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Where $E_2(f) = \frac{f^{(4)}(c)}{135}$

And Gauss-Legendre three-point rule is

$$\int_{-1}^{1} f(x)\,dx \cong G_3(f) = \frac{5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)}{9}$$

Where $E_3(f) = \frac{f^{(6)}(c)}{15{,}750}$

ملاحظة: اذا كان المطلوب ايجاد تكامل الدالة f على فترة *[a,b]* لاتساوي *[-1,1]* نستخدم change of variable

$$x = \frac{(b-a)t+(b+a)}{2}$$

$$dx = \frac{b-a}{2}dt$$

$$\int_{a}^{b} f(x)\,dx = \int_{-1}^{1} f\left(\frac{(b-a)t+(b+a)}{2}\right)\left(\frac{b-a}{2}\right)dt$$

$$= \frac{b-a}{2}\sum_{k=1}^{n} w_{n,k} f\left(\frac{(b-a)t_{n,k}+(b+a)}{2}\right)$$

Example(10): Use the two-point Gauss-Legendre rule to approximate the following integral

$$\int_{-1}^{1} \frac{dx}{x+2}$$

Solution:

$$\int_{-1}^{1} \frac{dx}{x+2} = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = f(-0.57735) + f(0.57735)$$

=0.70291+0.38800=1.09091

Where the exact solution is

$$\int_{-1}^{1} \frac{dx}{x+2} \ln(3) - \ln(1) \cong 1.09861$$

Example(11): Use the two-point Gauss-Legendre rule to approximate the following integral

$$\int_{0}^{2} e^{x^2} dx$$

Solution: To solve this integral we have change of variable then apply method

a=0, b=2

$$x = \frac{(b-a)t+(b+a)}{2} = \frac{(2-0)t+(2+0)}{2} = t+1$$

$$dx = \frac{b-a}{2}dt = \frac{2-0}{2}dt = dt$$

$$\int_{0}^{2} e^{x^2} dx = \frac{b-a}{2} \int_{-1}^{1} e^{(t+1)^2} dt = \int_{-1}^{1} e^{(t+1)^2} dt$$

$$\int_{0}^{2} e^{x^2} dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = e^{\left(-\frac{1}{\sqrt{3}}+1\right)^2} + e^{\left(\frac{1}{\sqrt{3}}+1\right)^2}$$

$$= 1.19558 + 12.0376 = 13.2332$$

Example(12): Use the three-point Gauss-Legendre rule to approximate the following integral $\int_0^2 e^{x^2} dx$, then use the Trapezoidal rule when n=1 and compare between the results.

Solution: H.W.

2.8 Newton-cotes Methods

Assume that $x_k = x_0 + kh$ are equally spaced nodes and $f_k = f(x_k)$. The first four closed Newton-Cotes quadrature formulas are

$$1 - \int_{x_0}^{x_1} f(x)dx \cong \frac{h}{2}[f_0 + f_1] \qquad \text{(trapezoidal rule)}$$

$$2 - \int_{x_0}^{x_2} f(x)dx \cong \frac{h}{3}[f_0 + 4f_1 + f_2] \qquad \text{(Simpson's 1/3 rule)}$$

$$3 - \int_{x_0}^{x_3} f(x)dx = \frac{3}{8}h[f_0 + 3f_1 + 3f_2 + f_3] \qquad \text{(Simpson's 3/8 rule)}$$

$$4 - \int_{x_0}^{x_4} f(x)dx = \frac{2}{45}h[7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4] \qquad \text{(Boole's rule)}$$

صيغة (١) هي صيغة شبه المنحرف والتي تتضمن نقطتين وخطأ البتر فيها من الرتبة $O(h^3)$, صيغة (٢) هي صيغة $1/3$ سمبسون وتتضمن ثلاث نقاط و صيغة (٣) هي صيغة $3/8$ سمبسون وتتضمن اربع نقاط ورتبة الخطأ في كلا الصيغتين $O(h^5)$, واما الصيغة (٤) فهي صيغة بول ذات النقاط الخمس وان رتبة الخطأ فيها $O(h^7)$.

Example (13): Find the value of the integral

$$\int_0^1 (1 + e^{-x}\sin(4x))\,dx$$

by using Newton-Cotes formulas.

Sol.:

For the trapezoidal rule, n=1 → h = 1 and

$$\int_0^1 f(x)dx \cong \frac{1}{2}[f(0) + f(1)] = \frac{1}{2}[1 + 0.72159] = 0.86079$$

For 1/3 Simpson's rule, n=2 → h = 1/2, and we get

$$\int_0^1 f(x)dx \cong \frac{1/2}{3}\left[f(0) + 4f(\frac{1}{2}) + f(1)\right] = \frac{1}{6}[1 + 4(1.55152) + 0.72159]$$

$$= 1.32128$$

For Simpson's 3/8 rule, n=3 →h = 1/3, and we obtain

$$\int_0^1 f(x)dx = \frac{3\left(\frac{1}{3}\right)}{8}\left[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1)\right]$$

$$= \frac{1}{8}[1 + 3(1.69642) + 3(1.23447) + 0.72159)]$$

$$= 1.31440$$

For Boole's rule, n=4→h = 1/4, and the result is

$$\int_0^1 f(x)dx = \frac{2\left(\frac{1}{4}\right)}{45}\left[7f(0) + 32f\left(\frac{1}{4}\right) + 12f\left(\frac{1}{2}\right) + 32f\left(\frac{3}{4}\right) + 7f(1)\right]$$

$$= \frac{1}{90}[(7(1) + 32(1.65534) + 12(1.55152) + 32(1.06666) + 7(0.72159)]$$

$$=1.30859$$

The true value of this integral is (1.3082506046426) and the approximation 1.30859 from Boole's rule is best.