Moment generating function (m.g.f)

الدالة المولدة العزوم

definition

The m.g.f of r.v. $\underline{X}$ define as

$$M_x(\underline{t}) = E\, e^{\underline{t}'\underline{X}}$$

$$\underline{X}\,(x_1, x_2 \dots x_p)\ ,\ \underline{t}\,(t_1, t_2 \dots t_p)$$

$$-h < t < h$$

Remark:

If $\underline{X} \sim N(\mu, \Sigma)$ then the m.g.f of a r.v. $\underline{X}$ defined as

$$M_{\underline{X}}(\underline{t}) = e^{\underline{t}'\mu + \frac{t'\Sigma t}{2}}$$

ملاحظة:

يمكن ايجاد العزوم ذات الرتبة (ر) حول الصفر لكل متغير من متغيرات المتجه العشوائي ($\underline{X}$) من الدالة المولدة للعزوم وكالاتي

$$E(\underline{X}) = \left.\frac{\partial M_{\underline{x}}(\underline{t})}{\partial t}\right|_{\underline{t}=0}$$

$$E(\underline{X}) = \begin{bmatrix} Ex_1 \\ Ex_2 \\ \vdots \\ Ex_p \end{bmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{pmatrix}$$

او يتم ايجاد مكونات متجه $E(\underline{X})$ كالاتي

$$E(X_1) = \frac{\partial M_X(t)}{\partial t_1}\Bigg|_{t=0}$$

$$E(X_2) = \frac{\partial M_X(t)}{\partial t_2}\Bigg|_{t=0}$$

$$\vdots$$

$$E(X_p) = \frac{\partial M_X(t)}{\partial t_p}\Bigg|_{t=0}$$

لايجاد العزم الثاني حول الصفر لكل متغير من متغيرات المتجه العشوائي $\underline{X}$:

$$E X_1^2 = \frac{\partial^2 M_X(t)}{\partial t_1^2}\Bigg|_{t=0}$$

$$E X_2^2 = \frac{\partial^2 M_X(t)}{\partial t_2}\Bigg|_{t=0}$$

$$\vdots$$

ولايجاد التباينات المشتركة بين متغيرين من متغيرات المتجه $\underline{X}$ نجد العزم المشترك بين المتغيرين $(X_i\, X_j)$ ويتم الحصول عليه من الصيغة الاتية :

$$E X_i X_j = \frac{\partial^2 M_X(t)}{\partial t_i\, \partial t_j}\Bigg|_{t=0}$$

٢ - $Cov(X_i, X_j) = E X_i X_j - E X_i\, E X_j$

Ex Consider $\underline{X} \sim N_2\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\right)$

① what is the m.g.f of $\underline{X}$

② use the m.g.f to find the vector mean and (var-cov matrix)

Solu

$$\mu = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

$$M_{\underline{X}}(\underline{t}) = e^{t'\mu + \frac{t'\Sigma t}{2}}$$

اولاً سوف نجد الاسس لـ e

$$\underline{t}'\mu = (t_1 \ t_2)\begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2t_1 + 3t_2$$

$$\underline{t}'\Sigma\underline{t} = (t_1 \ t_2)\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = 2t_1^2 + 3t_2^2 + 2t_1t_2$$

$$\therefore M_{\underline{X}}(\underline{t}) = e^{2t_1 + 3t_2 + \frac{1}{2}[2t_1^2 + 3t_2^2 + 2t_1t_2]}$$

$$= e^{2t_1 + 3t_2 + t_1^2 + \frac{3}{2}t_2^2 + t_1t_2}$$

$$E(X_1) = \left.\frac{\partial M_{\underline{X}}(\underline{t})}{\partial t_1}\right|_{t=0} = \left. e^{2t_1 + 3t_2 + t_1^2 + \frac{3}{2}t_2^2 + t_1t_2}(2 + 2t_1 + t_2)\right|_{\underline{t}=0}$$

$$= 2 = \mu_1$$

$$E(X_2) = \left.\frac{\partial M_{\underline{X}}(\underline{t})}{\partial t_2}\right|_{t=0}$$

$$E(X_2) = e^{2t_1+3t_2+t_1^2+\frac{3}{2}t_2^2+t_1t_2}(3+3t_2+t_1)\Big|_{t=0}$$

$$= 3 = \mu_2$$

$$V(X_1) = EX_1^2 - (EX_1)^2$$

$$EX_1^2 = \frac{\partial^2 M_X(t)}{\partial t_1^2}\Big|_{t=0}$$ مشتقة حاصل ضرب دالتين

$$= e^{2t_1+3t_2+t_1^2+t_1t_2+\frac{3}{2}t_2^2}(2) + (2+2t_1+t_2)(2+2t_1+t_2)\,e^{2t_1+3t_2+t^2+t_1t_2+\frac{3}{2}t_2^2}$$

$$\therefore E(X_1)^2 = (2) + (2)(2) = 6$$

$$\therefore V(X_1) = EX_1^2 - (EX_1)^2 = 6 - (2)^2 = \boxed{2}$$

$$EX_2^2 = \frac{\partial^2 M_X(t)}{\partial t_2^2}\Big|_{t=0}$$

$$= (3)\,e^{2t_1+3t_2+t_1^2+t_1t_2+\frac{3}{2}t_2^2} + (3+3t_2+t_1)(3+3t_2+t_1)\,e^{2t_1+3t_2+t_1^2+t_1t_2+\frac{3}{2}t_2^2}$$

$$\therefore EX_2^2 = (3) + (3)(3) = 12$$

$$\therefore V(X_2) = EX_2^2 - (E(X_2))^2 = 12 - (3)^2 = \boxed{3}$$

$v\text{-}cov(X_1\, X_2) = E X_1 X_2 - E X_1 E X_2$

$$E X_1 X_2 = \left.\frac{\partial^2 M_X(\underline{t})}{\partial t_1 \partial t_1}\right|_{t=0}$$

$$= (2+2t_1+t_2)(3+3t_2+t_1)\, e^{2t_1+3t_2+t_1^2+\frac{3}{2}t_2^2+t_1t_2} + (1)\, e^{2t_1+3t_2+t_1^2+\frac{3}{2}t^2+t_1t_2}$$

$E(X_1 X_2) = (2)(3)+1 = \boxed{7}$

$\therefore\ v\text{-}cov(X_1\, X_2) = \sigma_{12} = 7 - (3)(2) = \boxed{1}$

Ex

$$\Sigma = \begin{pmatrix} 6 & -1 \\ & 3 \end{pmatrix}, \quad \underline{\mu} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\underline{X} \sim N_2(\mu, \Sigma)$

① what is the m.g.f of $\underline{X}$

② use the m.g.f to find μ, {v-cov(X_1, X_2)

Notes:-

In the simplest form of m.g.f., we note that :

$\mu_1 \simeq$ Coefficient of t_1 only

$\mu_2 \simeq$ Coeff. of t_2 only

In general:-

$\mu_j \simeq$ Coeffi. of t_j only $\quad j = 1, 2, \dots, p$

$\sigma_1^2 \simeq 2 *$ Coeff. of t_1^2 only

$\sigma_2^2 \simeq 2 *$ Coeffi. of t_2^2 only

In general

$\sigma_j^2 \simeq 2 *$ Coffi. of t_j^2 only

$\sigma_{12} \simeq$ Coffi. of $t_1 t_2$ only

In general

$\sigma_{ij} \simeq$ Coeff of $t_i t_j$ only

Example:

Let $M_{\underline{x}}(\underline{t}) = e^{2t_1 - 2t_3 + 0.5t_1^2 + 3t_2^2 + 4t_3^2 - t_1 t_2 + 0.2t_1}$

Find: (1) $\underline{\mu}$, Σ, ρ_{13}

$\mu_1 = 2$, $\mu_2 = 0$ $\mu_3 = -2$

$$\therefore \underline{\mu} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

$$\sigma_1^2 = 2 \times \frac{1}{2} = 1$$

$$\sigma_2^2 = 2 \times 3 = 6$$

$$\sigma_3^2 = 2 \times 4 = 8$$

$$\sigma_{12} = -1 \quad , \quad \sigma_{13} = 0.25 \quad , \sigma_{23} = 0$$

$$\therefore \Sigma = \begin{pmatrix} 1 & -1 & 0.25 \\ & 6 & 0 \\ & & 8 \end{pmatrix}$$

$$\rho_{13} = \frac{\sigma_{13}}{\sigma_1 \sigma_2} = \frac{0.25}{\sqrt{1}\sqrt{8}}$$

بعض خصائص الدالة المولدة للعزوم في حالة متجه المتغيرات

(1) اذا كانت المتجه العشوائي (X) ذات بعد (p×1) له دالة مولدة للعزوم $M_X(t)$ واذا عرفنا متجه عشوائي جديد $Y = X + d$ حيث ان المتجه d هو متجه ثوابت ذات بعد (p×1) فان الدالة المولدة للعزوم لـ (Y) هي

$$M_Y(t) = e^{t'd} M_X(t)$$

Proof.

$$M_Y(t) = E e^{t'Y} = E e^{t'(X+d)} = E e^{t'X + t'd}$$

$$= e^{t'd} E e^{t'X} = e^{t'd} M_X(t)$$

(2) لكن $Z = CX$

حيث ان (C) هي مصفوفة ثوابت وابعادها متوافقة مع المتجه X بشرط ان لا تزيد عدد صفوفها عن (p) فان الدالة المولدة للعزوم لـ (Z) هي

$$M_Z(t) = M_X(C't)$$

Proof.

$$M_Z(t) = E e^{t'Z} \qquad E e^{t'CX} \qquad (t^*)' = t'C$$

$$t^* = C't$$

$$\therefore M_Z(t) = E e^{t^{*\prime} X} = M_X(t^*) = M_X(C't)$$

Linearity Property in multivariate normal distribution

اذكر واثبت

Using m.g.f

Theorem نظرية

Let $\underline{X} \sim N(\underline{\mu}, \Sigma)$, and if C is a $(p \times p)$ matrix of constants and non-singular then the Linear function $\underline{Y} = C\underline{X} \sim N_p(C\underline{\mu}, C\Sigma C')$

Proof:

$\underline{Y} = C\underline{X}$ $\therefore M_{\underline{Y}}(\underline{t}) = E\,e^{\underline{t}'\underline{Y}} = E\,e^{\underline{t}'C\underline{X}}$

$= E\,e^{\underline{t}^{*\prime}\underline{X}}$ such that $\underline{t}^{*\prime} = \underline{t}'C$

$\therefore M_{\underline{Y}}(\underline{t}) = M_{\underline{X}}(\underline{t}^{*})$ --- (*)

but $\underline{X} \sim N(\underline{\mu}, \Sigma)$ $\qquad t^{*\prime} = t'C$

$\qquad t^{*} = C't$

$\therefore M_{\underline{X}}(\underline{t}^{*}) = e^{t^{*\prime}\mu + \frac{1}{2} t^{*\prime}\Sigma t^{*}}$

$= e^{t'C\mu + \frac{1}{2} t'C\Sigma C't}$ --- **

نعوض المعادلة (**) في المعادلة (*) ينتج ان

$M_{\underline{Y}}(\underline{t}) = e^{t'C\mu + \frac{1}{2} t'C\Sigma C't}$

$\therefore \underline{Y} \sim N(C\mu, C\Sigma C')$

$\therefore f(\underline{Y}) = \dfrac{1}{(2\pi)^{p/2}\,|C\Sigma C'|^{1/2}}\, e^{-\frac{1}{2}(\underline{Y} - C\mu)'(C\Sigma C')^{-1}(\underline{Y} - C\mu)}$

Example.

Let $\underline{X} \sim N_2(\underline{\mu}, \Sigma)$, use the m.g.f to find the p.d.f of the r.v. $\underline{Y}$ such that

$$\underline{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{pmatrix} 2X_1 \\ X_1 + X_2 \end{pmatrix}$$

Solu:

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = C\,\underline{X}$$

$$M_{\underline{Y}}(\underline{t}) = E e^{\underline{t}'\underline{Y}} = E e^{\underline{t}'C\underline{X}} = E e^{\underline{t}^{*'}\underline{X}} = M_{\underline{X}}(\underline{t}^*)$$

$$= e^{\underline{t}'C\mu + \frac{1}{2}\underline{t}'C\Sigma C'\underline{t}}$$

$$\therefore Y \sim N_2(C\mu, C\Sigma C')$$

$$\therefore f(\underline{Y}) = \frac{1}{(2\pi)|C\Sigma C'|^{\frac{1}{2}}} e^{-\frac{1}{2}(\underline{Y} - C\mu)'(C\Sigma C')^{-1}(Y - C\mu)}$$

$$E\underline{Y} = C\underline{\mu} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 2\mu_1 \\ \mu_1 + \mu_2 \end{bmatrix}$$

$$\text{v-cov}\,\underline{Y} = C\Sigma C'$$

$$= \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ & \sigma_2^2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

H.W.

Let $\underline{X} \sim N_2\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{bmatrix} 1 & -1 \\ & 2 \end{bmatrix}\right]$ use the m.g.f to find the p.d.f of the r.v. $\underline{Y}$

$$\underline{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 2X_1 \\ X_1 + X_2 \end{bmatrix}$$

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Let $\underline{X} \sim N_3\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ & 2 & -1 \\ & & 3 \end{bmatrix}\right)$ use the m.g.f to find the p.d.f of r.v. $\underline{Y}$

$$Y = \begin{bmatrix} 2X_1 + X_2 - X_3 \\ X_2 \\ X_1 + X_3 \end{bmatrix}$$