(3) LU Factorization Method

3-1 Doolittle Factorization

To solve

$$A_{n\times n}\, X_{n\times 1} = B_{n\times 1} \qquad \ldots\ldots \ (1)$$

by LU factorization method, this method is one of the direct methods, as it converts the coefficient matrix A into the product of two matrices of the same order: L (a lower triangular matrix) and U (an upper triangular matrix).

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \ell_{21} & 1 & 0 & 0 & \cdots & 0 & 0 \\ \ell_{31} & \ell_{32} & 1 & 0 & \cdots & 0 & 0 \\ \ell_{41} & \ell_{42} & \ell_{43} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & 1 & 0 \\ \ell_{n1} & \ell_{n2} & \ell_{n3} & \ell_{n4} & \cdots & \ell_{n,n-1} & 1 \end{bmatrix}_{n\times n} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} & \cdots & u_{1,n-1} & u_{1n} \\ 0 & u_{22} & u_{23} & u_{24} & \cdots & u_{2,n-1} & u_{2n} \\ 0 & 0 & u_{33} & u_{34} & \cdots & u_{3,n-1} & u_{3n} \\ 0 & 0 & 0 & u_{44} & \cdots & u_{4,n-1} & u_{4n} \\ \vdots & \vdots & \vdots & \vdots & \cdots & u_{n-1,n-1} & u_{n-1,n} \\ 0 & 0 & 0 & 0 & \cdots & 0 & u_{n,n} \end{bmatrix}_{n\times n}$$

Then we solve two triangular systems

LY=B for Y and UX=Y for X(solution) Where $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$.

By using the Gauss elimination method without the partial pivoting process to factorize a matrix A into a product of two matrices L and U.

$$\ell_{ij} = \begin{Bmatrix} 0 & if & i < j \\ 1 & if & i = j \\ m_{ij} & if & i > j \end{Bmatrix} \quad \text{and} \quad u_{ij} = \begin{Bmatrix} r_{ij} & if & i < j \\ & & \\ 0 & if & i > j \end{Bmatrix}$$

$\Rightarrow$ A = L.U

Example 6: Factorization the matrix A into L.U by using Doolittle Factorization.

$$A = \begin{bmatrix} 4 & -9 & 2 \\ 2 & -4 & 6 \\ 1 & -1 & 3 \end{bmatrix}$$

Solution: After using the elimination process in the Gaussian elimination method, we obtain an upper triangular matrix, which is *U*:

$$U = \begin{bmatrix} 4 & -9 & 2 \\ 0 & 0.5 & 5 \\ 0 & 0 & -10 \end{bmatrix}$$

and the matrix L is

$$L = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix}, \quad \text{where } m_{21} = \frac{2}{4} = 0.5 \ , \ m_{31} = \frac{1}{4} = 0.25 \ , \ m_{32} = \frac{1.25}{0.5} = 2.5$$

$$A = \begin{bmatrix} 4 & -9 & 2 \\ 2 & -4 & 6 \\ 1 & -1 & 3 \end{bmatrix} = L.U = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.25 & 2.5 & 1 \end{bmatrix} . \begin{bmatrix} 4 & -9 & 2 \\ 0 & 0.5 & 5 \\ 0 & 0 & -10 \end{bmatrix}$$

Example 7: H.W.

Use LU factorization to solve

$3x_1+3x_3=0$
$-x_2+3x_3=1$
$x_1+3x_2=2$

Exact solution is $x_1 = -\frac{5}{8}, \ x_2 = \frac{7}{8}, \ x_3 = \frac{5}{8}$