A Non-linear Programming Problem (NLPP)

Assume that we want to maximize or minimize a function $f(x_1, x_2, \ldots, x_n)$ utility function that we want to maximize or a cost function that we want to minimize, subject to the constraints $g_1(x_1, x_2, \ldots, x_n) = 0$, $g_2(x_1, x_2, \ldots, x_n) = 0$ and $g_m(x_1, x_2, \ldots, x_n) = 0$. Suppose we want to:

$$\left.\begin{array}{ll} Max & f(x_1, x_2, \ldots, x_n) \\ s.t. & g_1(x_1, x_2, \ldots, x_n) = 0 \\ & g_2(x_1, x_2, \ldots, x_n) = 0 \\ & . \\ & . \\ & g_m(x_1, x_2, \ldots, x_n) = 0 \end{array}\right\} \qquad (I)$$

Here : $f.g : i = 1, 2, 3, \ldots, m$ (may be linear or non-linear).

Methods to Solve Non-linear Programming Problem

1. Calculus:

How to solve the non linear programming problem NLPP, by use of **Calculus**. We know from calculus that if we want to find the stationary values (max or min) of the function $f(x_1, x_2)$, in some cases one can be solve for x_2 as a function of x_1 and then find the min or max point of a one variable function. That is:

$$\frac{\partial f}{\partial x_1} = 0 \ .$$

Example : Find the Maximum value of the NLPP:

$$\begin{array}{ll} Max & f(x_1, x_2) = -4 - 3(1 - x_1)^2 - (1 - x_2)^2 \\ s.t. & 3x_1 + x_2 = 5 \end{array}$$

Solution :

From constraint $3x_1 + x_2 = 5$, we get:

$$x_2 = 5 - 3x_1 \qquad \text{..........(1)}$$

Putting (1) in objective function we get:

$$\begin{aligned} Max \;\; f(x_1, x_2) &= -4 - 3(1 - x_1)^2 - (1 - 5 + 3x_1)^2 \\ &= -4 - 3(1 - x_1)^2 - (-4 + 3x_1)^2 \end{aligned} \qquad \text{..........(2)}$$

The gradient of x_1 we have :

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= -6(1 - x_1)(-1) - 2(-4 + 3x_1)(3) \\ &= 6(1 - x_1) - 6(-4 + 3x_1) \\ &= 30 - 24x_1 \end{aligned} \qquad \text{..........(3)}$$

Then gradient equal zero, we get:

$$0 = 30 - 24x_1 \qquad \text{..........(4)}$$

From above equation we have:

$$30 = 24x_1 \Rightarrow x_1 = \frac{30}{24} = \frac{5}{4} \qquad \text{..........(5)}$$

Putting (5) in (1), we get :

$$x_2 = 5 - 3x_1 = 5 - 3\left(5/4\right) = \frac{20 - 15}{4} = \frac{5}{4} \qquad \text{..........(6)}$$

Now, putting (5) and (6) in objective function we get:

$$f = -4.25$$

Example : Find the Minimum value of the NLPP:

$$\begin{aligned} Max \quad & f(x,y) = x^2 y - \ln(x) \\ s.t. \quad & 8x + 3y = 0 \end{aligned}$$

Solution :

We solve constraint $8x + 3y = 0$, we get:

$$y = -\frac{8}{3}x \qquad \text{..........}(1)$$

Putting (1) in objective function, we get:

$$Min \;\; f(x,y) = -\frac{8}{3}x^3 - \ln(x) \qquad \text{..........}(2)$$

The gradient we have:

$$\frac{\partial f}{\partial x_1} = -8x^2 - \frac{1}{x} \qquad \text{..........}(3)$$

Then gradient equal zero, we get:

$$0 = -8x^2 - \frac{1}{x} \qquad \text{..........}(4)$$

From above equation we have:

$$\begin{aligned} & -8x^2 = \frac{1}{x} \Rightarrow -8x^3 = 1 \\ \Rightarrow \;\; & x^3 = -\frac{1}{8} \Rightarrow x = -\frac{1}{2} \end{aligned} \qquad \text{..........}(5)$$

From (5) and (1), we have:

$$y = -\frac{8}{3}x = -\frac{8}{3}\left[-\frac{1}{2}\right] = \frac{4}{3}. \qquad \text{..........}(6)$$

The minimum point is :

$$x = -\frac{1}{2} \quad , \quad y = \frac{4}{3} . \qquad \text{..........(7)}$$