Directional Derivatives and Gradients

المشتقات الاتجاهية والتدرجات

Definitions and Theorems: **تعاريف ونظريات**

- Let $z = f(x,y)$ be a function whose partial derivatives exist. The gradient of f is the vector :

  ليكن $z = f(x,y)$ دالة لها مشتقات جزئية موجودة. فأن التدرج f هو المتجه :

$$grad f(x,y) = \nabla f(x,y) = \langle f_x\,,f_y\rangle = f_x(x,y)i + f_y(x,y)j$$

- Let u be a unit vector in the plane, and let f be a differentiable function of x and y. the directional derivative of f in the direction of u is:

  ليكن u متجه وحدة في المستوى، وليكن f دالة قابلة للاشتقاق لـ x و y. فأن المشتقة الاتجاهية لـ f في اتجاه u هي:

$$D_u f(x,y) = \nabla f(x,y).u$$

- $D_u f(x,y) = \|\nabla f(x,y)\|.\cos\theta$ where θ is the angle between the gradient and the unit vector u. the directional derivative is a maximum when $\cos\theta = 1$ and mimimum when $\cos\theta = -1$

لإيجاد اكبر قيمة ممكنة maximum للمشتقات الاتجاهية يتم من خلال استخدام القانون أعلاه ولكن عندما تكون قيمة الزاوية $\theta = 0$ في هذه الحالة $\cos\theta = 1$ ويصبح القانون كالاتي:

$$maxD_u f(x,y) = \|\nabla f(x,y)\|$$

ولإيجاد اقل قيمة ممكنة mimimum للمشتقات الاتجاهية يتم من خلال استخدام القانون أعلاه ولكن عندما تكون قيمة الزاوية $\theta = 180$ في هذه الحالة $\cos\theta = -1$ ويصبح القانون كالاتي:

$$minD_u f(x,y) = -\|\nabla f(x,y)\|$$

Example (1): Find the gradient of the function $f(x,y) = ylnx + xy^2$ at the point (1,2).

Solve:

$$\nabla f(x,y) = \langle f_x\,,f_y\rangle = \langle\frac{y}{x} + y^2\,,lnx + 2xy\rangle$$

$$\nabla f(1,2) = \langle\frac{2}{1} + (2)^2\,,ln1 + 2(1)(2)\rangle = \langle 2+4\,,0+4\rangle = \langle 6\,,4\rangle = 6i + 4j$$

Example (2): Find the gradient of the function $f(x,y,z) = x^2 + y^2 - 4z$ at the point (2,-1,1).

Solve:

$$\nabla f(x,y,z) = \langle f_x\,,f_y\,,f_z\rangle = \langle 2x\,,2y,-4\rangle$$

$$\nabla f(2,-1,1) = \langle 2(2)\,,2(-1),-4\rangle = \langle 4,-2,-4\rangle = 4i - 2j - 4k$$

Example (3): Find the directional derivative of the function $f(x,y) = 4 - x^2 - \frac{1}{4}y^2$ at the point (1,2) in the direction of $u = \left(\cos\frac{\pi}{3}\right)i + \left(\sin\frac{\pi}{3}\right)j$

Solve:

$$D_u f(x,y) = \nabla f(x,y).u$$

$$\nabla f(x,y) = \langle f_x \,, f_y \rangle = \langle -2x \,, -\frac{1}{2}y \rangle$$

$$\nabla f(1,2) = \langle -2(1), -\frac{1}{2}(2) \rangle = \langle -2, -1 \rangle$$

$$D_u f(x,y) = \langle -2, -1 \rangle . \langle cos\frac{\pi}{3}, sin\frac{\pi}{3} \rangle$$

$$D_u f(x,y) = (-2)\left(cos\frac{\pi}{3}\right) + (-1)\left(sin\frac{\pi}{3}\right)$$

$$= (-2)(0.5) + (-1)(0.866) = -1 - 0.866 = -1.866$$

Example (4): Find the directional derivative of the function $f(x,y,z) = x^2 + y^2 + z^2$ at the point (1,1,1) in the direction of $u = i - j + k$

Solve:

$$D_u f(x,y,z) = \nabla f(x,y,z).u$$

$$\nabla f(x,y,z) = \langle f_x \,, f_y, f_z \rangle = \langle 2x, 2y, 2z \rangle$$

$$\nabla f(1,1,1) = \langle 2(1), 2(1), 2(1) \rangle = \langle 2,2,2 \rangle$$

$$D_u f(x,y,z) = \langle 2,2,2 \rangle . \langle 1, -1, 1 \rangle$$

$$D_u f(x,y,z) = (2)(1) + (2)(-1) + (2)(1) = 2 - 2 + 2 = 2$$

Example (5): Find the maximum value of the directional derivative of the function $f(x,y) = x^2 + 2xy$ at the point (0,1) .

Solve:

$$maxD_u f(x,y) = \|\nabla f(x,y)\|$$

$$\nabla f(x,y) = \langle f_x \,, f_y \rangle = \langle 2x + 2y \,, 2x \rangle$$

$$\nabla f(0,1) = \langle \big(2(0) + 2(1)\big) \,, 2(0) \rangle = \langle 2 \,, 0 \rangle$$

$$maxD_u f(x,y) = \|\nabla f(0,1)\| = \sqrt{2^2 + 0^2} = \sqrt{4} = 2$$

Example (5): Find the minimum value of the directional derivative of the function $f(x,y,z) = xy^2z^2$ at the point (2,1,1) .

Solve:

$$minD_u f(x,y,z) = -\|\nabla f(x,y,z)\|$$

$$\nabla f(x,y,z) = \langle f_x \,, f_y, f_z \rangle = \langle y^2z^2 \,, 2xyz^2 \,, 2xy^2z \rangle$$

$$\nabla f(2,1,1) = \langle (1)^2(1)^2 \,, 2(2)(1)(1)^2 \,, 2(2)(1)^2(1) \rangle = \langle 1,4 \,, 4 \rangle$$

$$minD_u f(x,y,z) = -\|\nabla f(2,1,1)\| = -\sqrt{1^2 + 4^2 + 4^2} = -\sqrt{1 + 16 + 16} = -\sqrt{33} = -5.74$$