Theorem : Dual of the dual is the primal itself.

Proof :

The primal Lpp is :

$$Min\ Z = \sum_{j=1}^{n} c_j\, x_j$$

$$S.t.: \sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad , \; i = 1 , 2 , 3 , \; . \; . \; . \; , m \qquad \text{..........(1)}$$

$$x_j \geq 0 \quad , \; j = 1 , 2 , 3 , \; . \; . \; . \; , n$$

Its dual is :

$$Max\ u = \sum_{i=1}^{m} b_i\, y_i$$

$$S.t.: \sum_{i=1}^{m} a_{ij}\, y_i \leq c_j \quad , \; j = 1 , 2 , 3 , \; . \; . \; . \; , n \qquad \text{..........(2)}$$

$$y_i \geq 0 \quad , \; i = 1 , 2 , 3 , \; . \; . \; . \; , m$$

We put this in the standard form of the primal :

$$-u = Min\ w = \sum_{i=1}^{m} - b_i\, y_i$$

$$S.t.: - \sum_{i=1}^{m} a_{ij}\, y_i \geq - c_j \, , \; j = 1 , 2 , 3 , \; . \; . \; . \; , n \qquad \text{..........(3)}$$

$$y_i \geq 0 \quad , \; i = 1 , 2 , 3 , \; . \; . \; . \; , m$$

The dual of this is :

$$Max\ u = \sum_{j=1}^{n} - c_j\, x_j$$

$$S.t.: - \sum_{j=1}^{n} a_{ij} x_j \leq - b_i \quad , \; i = 1 , 2 , \; . \; . \; . \; , m \qquad \text{..........(4)}$$

$$x_j \geq 0 \quad , \; j = 1 , 2 , 3 , \; . \; . \; . \; , n$$

which is written as :

$$Min\ u' = \sum_{j=1}^{n} c_j\, x_j$$

$$S.t.: \sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad , \; i = 1 , 2 , 3 , \; . \; . \; . \; , m \qquad \text{..........(5)}$$

$$x_j \geq 0 \quad , \; j = 1 , 2 , 3 , \; . \; . \; . \; , n$$

which is primal.

Theorem : The value of the objective function z for any feasible solution of the primal :

$$Min\ Z\ = \sum_{j=1}^{n} c_j\ x_j$$

$$S.t.: \sum_{j=1}^{n} a_{i\,j} x_j\ \geq b_i\ \ ,\ \ i=1\,,\,2\,,\,3\,,\ \ldots\ ,\,m \qquad \ldots\ldots\ldots.(I)$$

$$x_j\ \geq 0\ \ ,\ \ j=1\,,\,2\,,\,3\,,\ \ldots\ ,\,n$$

Primal is not lees than the value of the objective function V any feasible solution dual :

$$Max\,V\ = \sum_{i=1}^{m} b_i\ y_i$$

$$S.t.: \sum_{i=1}^{m} a_{i\,j}\ y_i\ \leq c_j\ \ ,\ \ j=1\,,\,2\,,\,3\,,\ \ldots\ ,\,n \qquad \ldots\ldots\ldots.(II)$$

$$y_i\ \geq 0\ \ ,\ \ i=1\,,\,2\,,\,3\,,\ \ldots\ ,\,m$$

Proof :

Put (I) and (II) in the standard form :

$$\begin{aligned}
Min\ Z\ &= c_1\,x_1 + c_2\,x_2 + \ldots\ \ldots + c_n\,x_n \\
S.t.:\quad & a_{11}x_1 + a_{12}x_2 + \ldots\ \ldots + a_{1n}x_n - x_{n+1} = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots\ \ldots + a_{2n}x_n - x_{n+2} = b_2 \\
& \ . \\
& \ . \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ldots\ldots\ldots.(I^*) \\
& \ . \\
& a_{m1}x_1 + a_{m2}x_2 + \ldots\ \ldots + a_{mn}x_n - x_{n+m} = b_m \\
& x_1\ ,\ x_2\ ,\ \ldots\ldots,\ x_n\ ,\ x_{n+1}\ \ldots\ .\ .,\ x_{n+m} \geq 0
\end{aligned}$$

Also,

$$\begin{aligned}
MaxV\ &= b_1 y_1 + b_2\,y_2 + \ldots\ \ldots + b_n\,y_n \\
S.t.:\quad & a_{11}y_1 + a_{21}y_2 + \ldots\ \ldots + a_{m1}y_m - y_{m+1} = c_1 \\
& a_{12}y_1 + a_{22}y_2 + \ldots\ \ldots + a_{m2}y_m - y_{m+2} = c_2 \\
& \ . \\
& \ . \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ldots\ldots\ldots.(II^*) \\
& \ . \\
& a_{1n}y_1 + a_{2n}y_2 + \ldots\ \ldots + a_{nm}y_m - y_{m+n} = c_n \\
& y_1\ ,\ y_2\ ,\ \ldots\ldots,\ y_m\ ,\ y_{m+1}\ \ldots\ .\ .,\ y_{m+n} \geq 0
\end{aligned}$$

- Multiply the primal constraint by $(y_1\ ,\ y_2\ ,\ \ldots\ldots,\ y_m)$ and add.
- Multiply the dual constraint by $(x_1\ ,\ x_2\ ,\ \ldots\ldots,\ x_n)$ and add.

- If we want to find (I^*) - (II^*) we can see that the 1-st row of the primal cancels the 1-st column of the dual, then the $m*n$ rows of the primal cancels with $m*n$ column of the dual.

The remainder is :

$$(-y_1x_{n+1} - y_2 x_{n+1} - \ldots\ldots - y_m x_{n+m}) - (x_1 y_{m+1} + x_2 y_{m+1} + \ldots\ldots + x_n y_{m+n})$$
$$= (b_1 y_1 + b_2 y_2 + \ldots\ldots + b_m y_m) - (c_1 x_1 + c_2 x_2 + \ldots\ldots + c_n x_n) = V - Z$$

Multiply both side by (-1) we have :

$$Z - V = y_1 x_{n+1} + y_2 x_{n+1} + \ldots\ldots + y_m x_{n+m} + x_1 y_{m+1} + x_2 y_{m+1} + \ldots\ldots + x_n y_{m+n}$$

Then :

$$Z - V \geq 0 \quad \Rightarrow \quad Min \ Z \geq Max \ V .$$

If a standard linear programming problem is bounded feasible, then so is its dual, their values are equal, and there exists optimal vectors for both problems.

<u>ملاحظات مهمة:</u>

$primal \leftarrow Min \ z$, $dual \leftarrow Max \ v$.

$m \leftarrow dual$ (نستخدم y وحدودها من (1)الى (m)).

$n \leftarrow primal$ (نستخدم x وحدودها من (1)الى (n)).

اذا كانت العلاقة $(\geq)$ (اكبر او يساوي) نطرح متغيرات خاملة.

اذا كانت العلاقة $(\leq)$ (اقل او يساوي) نضيف متغيرات خاملة.

دائماً علاقة $(\geq)$ مع Min , $(\leq)$ مع Max.