Kuhn-Tucker conditions

Derive the Kuhn-Tucker conditions for NLPP:

صيغة رياضية في اشتقاق شروط Kuhn-Tucker لحل مسائل في البرمجة اللاخطية ولسهولة الاشتقاق نأخذ متغيرين معقيد واحد فقط ومن ثم نعمم الى x_n ,....., x_2 , x_1 وقيود.

البداية:

$$\begin{aligned} &Max \;\; f(x_1, x_2) \\ &s.t. \;\; g(x_1, x_2) \geq 0 \end{aligned} \qquad \text{NLPP}$$

Solution:

We may write:

$$\begin{aligned} &Max \;\; f(x_1, x_2) \\ &s.t. \;\; g(x_1, x_2) + z^2 = 0 \end{aligned}$$

Using the lagranges multipliers method we have:

$$h(x_1, x_2, z^2, \lambda) = f(x_1, x_2) - \lambda\,(g(x_1, x_2) + z^2)$$

Now,

$$h(x_1, x_2, z^2, \lambda) = f(x_1, x_2) - \lambda\,(g(x_1, x_2) + z^2)$$

Then:

$$\frac{\partial h}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial g}{\partial x_1} = 0 \qquad \text{..........}(1)$$

.

$$\frac{\partial h}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda \frac{\partial g}{\partial x_2} = 0 \qquad \text{..........}(2)$$

.

$$\frac{\partial h}{\partial z} = -2\lambda z = 0 \qquad \text{..........} (3)$$

.

$$\frac{\partial h}{\partial \lambda} = -[g(x_1, x_2) + z^2] = 0 \qquad \text{..........} (4)$$

Now, we want to find z^2 ? . From eq (4) we have:

$$z^2 = -g(x_1, x_2) \qquad \text{..........} (5)$$

From eq (3) we have :

$$\lambda z = 0 \quad \Rightarrow \quad \lambda z^2 = 0 \qquad [\text{بالضرب في } z]$$

$$\Rightarrow \lambda(-g(x_1, x_2)) = 0$$

بالتعويض :

$$\Rightarrow \quad \lambda g(x_1, x_2) = 0 \qquad \text{..........} (6)$$

So, if we are going to find:

$$\begin{aligned} &Max \;\; f(x_1, x_2) \\ &s.t. \;\; g(x_1, x_2) \geq 0 \end{aligned} \qquad \text{NLPP}$$

Then, the stationary points are given by:

$$\left.\begin{aligned} \frac{\partial f}{\partial x_1} - \lambda \frac{\partial g}{\partial x_1} &= 0 \\ \frac{\partial f}{\partial x_2} - \lambda \frac{\partial g}{\partial x_2} &= 0 \\ \lambda g(x_1, x_2) &= 0 \\ g(x_1, x_2) &\geq 0 \end{aligned}\right\}$$

these conditions are called Kuhn-Tacker condition $\lambda \geq 0$.

<u>Ingeneral</u> :

If we have an optimization problem :

$$
\begin{aligned}
&Max\ (\min)\ f(x_1, x_2, \ldots, x_n) \\
&s.t. \quad g_1(x_1, x_2, \ldots, x_n) \geq 0 \\
&\qquad\quad g_2(x_1, x_2, \ldots, x_n) \geq 0 \\
&\qquad\quad . \\
&\qquad\quad . \\
&\qquad\quad g_m(x_1, x_2, \ldots, x_n) \geq 0
\end{aligned}
$$

Then, the stationary points are given by:

$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial g_1}{\partial x_1} - \ldots - \lambda_m \frac{\partial g_m}{\partial x_1} = 0$$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial g_1}{\partial x_2} - \ldots - \lambda_m \frac{\partial g_m}{\partial x_2} = 0$$

$$\frac{\partial f}{\partial x_n} - \lambda_1 \frac{\partial g_1}{\partial x_n} - \ldots - \lambda_m \frac{\partial g_m}{\partial x_n} = 0$$

With,

$$
\begin{aligned}
&\lambda_1\, g_1(x_1, \ldots., x_n) = 0 \\
&\lambda_2\, g_2(x_1, \ldots., x_n) = 0 \\
&. \\
&. \\
&\lambda_m\, g_m(x_1, \ldots., x_n) = 0
\end{aligned}
$$

And

$$
\begin{aligned}
&g_1(x_1, \ldots., x_n) \geq 0 \\
&g_2(x_1, \ldots., x_n) \geq 0 \\
&. \\
&. \\
&g_m(x_1, \ldots., x_n) \geq 0
\end{aligned}
$$

and,

$$\lambda_1 \geq 0\ ,\ \lambda_2 \geq 0\ , \ldots\ldots\ldots\ldots, \ \lambda_m \geq 0.$$

Example :

Use Kuhn-Tucker conditions to solve:

$$Max \quad f(x_1, x_2) = 2x_1 + x_1 x_2 + 3x_2$$
$$s.t. \qquad x_1^2 + x_2 \geq 3$$

Solution :

Our $g(x_1, x_2) = x_1^2 + x_2 - 3 \geq 0$. The Kuhn-Tacker condition are:

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial g}{\partial x_1} = 0 \quad \Rightarrow 2 + x_2 - \lambda(2x_1) = 0 \qquad \ldots\ldots\ldots\ldots(1)$$

$$\frac{\partial f}{\partial x_2} - \lambda \frac{\partial g}{\partial x_2} = 0 \quad \Rightarrow x_1 + 3 - \lambda(1) = 0 \qquad \ldots\ldots\ldots\ldots(2)$$

معادلات التأكيد

$$\lambda g(x_1, x_2) = 0 \quad \Rightarrow \lambda(x_1^2 + x_2 - 3) = 0 \qquad \ldots\ldots\ldots\ldots(3)$$

$$g(x_1, x_2) \geq 0 \quad \Rightarrow x_1^2 + x_2 - 3 \geq 0 \qquad \ldots\ldots\ldots\ldots(4)$$

$$\lambda \geq 0 \quad \Rightarrow \lambda \geq 0 \qquad \ldots\ldots\ldots\ldots(5)$$

From (2) we get:

$$\lambda = x_1 + 3 \qquad \ldots\ldots\ldots\ldots(6)$$

Write (1) and (6):

$$2 + x_2 - (x_1 + 3)(2x_1) = 0 \quad \Rightarrow \quad 2 + x_2 - 2x_1^2 - 6x_1 = 0$$
$$\Rightarrow \quad 2x_1^2 + 6x_1 - x_2 - 2 = 0 \qquad \ldots\ldots\ldots\ldots(7)$$

and

$$(x_1 + 3)(x_1^2 + x_2 - 3) = 0 \qquad \ldots\ldots\ldots\ldots(8)$$

from (8) we get:

$$x_1 = -3 \qquad or \qquad x_2 = 3 - x_1^2 \qquad \ldots\ldots\ldots\ldots(9)$$

if $x_1 = -3$ we have,

$$x_2 = -2 \qquad \text{from (7)}$$

from (6), we have:

$$\lambda = 0$$

Then :

$$x_1 = -3 \quad , \quad x_2 = -2 \quad , \quad \lambda = 0$$ satisfies the K.T.C

لاحظ إن هذه النتيجة تحقق المعادلات (4) و (5)

The other possibility:

If :

$$x_2 = 3 - x_1^2$$

Putting in (7) we have:

$$x_1 = \frac{-6 \mp \sqrt{36+60}}{6} \quad \Rightarrow \quad x_1 = -1 \mp \frac{2\sqrt{6}}{3}$$

Then:

$$\therefore \quad x_2 = -\frac{2}{3} \mp \frac{4\sqrt{6}}{3}$$

From (6), we get:

$$\lambda \quad = -1 \mp \frac{2\sqrt{6}}{3} + 3 = \quad 2 + 1.6 \quad > 0$$

$$g(x_1, x_2) = x_1^2 + x_2 - 3$$

$$= (-1 \mp \frac{2\sqrt{6}}{3}) + (-\frac{2}{3} \mp \frac{4\sqrt{6}}{3}) - 3 = 0$$

∴ تحقق الشرط الاخير، لدينا الان مجموعتان من الحلول وهما:

مجموعة (1) $x_1 = -3 \quad , \quad x_2 = -2 \quad , \quad \lambda = 0$

مجموعة (2) $x_1 = -1 \mp \frac{2\sqrt{6}}{3} \; , \; x_2 = -\frac{2}{3} \mp \frac{4\sqrt{6}}{3} \; , \; \lambda > 0$

وكلاهما يحققان الشرطين (4) و (5) فايهما نختار:

$$Max \;\; f(x_1, x_2) = 2x_1 + x_1 x_2 + 3x_2$$

$$f = -6 \;\; Min$$

$$f = 10.6 \;\; Max$$