Differential equation of first order and first degree

First-order and first-degree differential equations generally take one of two forms:

$$\frac{dy}{dx} = F(x, y) \qquad or \qquad M(x, y)dx + N(x, y)dy = 0$$

تأخذ المعادلات التفاضلية من الرتبة الاولى والدرجة الاولى بشكل عام احدى الشكلين

There are several types of first-order and first-degree differential equations:

1- Equations whose variables are separated. المعادلات التي تنفصل متغيراتها
2- linear differential equation. المعادلة التفاضلية الخطية
3- Complete differential equation. المعادلة التفاضلية التامة
4- Non-Complete differential equation. المعادلة التفاضلية غير التامة
5- Homogenous differential equation. المعادلة التفاضلية المتجانسة

linear differential equation

These are the equations that lead to the following form:

$$\frac{dy}{dx} + p(x)y = Q(x) \qquad or \qquad \frac{dx}{dy} + p(y)x = Q(y)$$

The linear differential equation is solved as follows:

1) Estimate the Integration Factor $I.F = e^{\int p(x)dx}$
2) Write the final form of the general solution of the differential equation $y.(I.F) = \int Q(x).(I.F)dx + c$

Example (1): Compute the general solution of differential equation

$$\frac{dy}{dx} + \frac{y}{x} = x^2 - 1$$

Solve:

$$p(x) = \frac{1}{x} \quad , \quad Q(x) = x^2 - 1$$

$$I.F = e^{\int p(x)dx}$$

$$I.F = e^{\int \frac{1}{x}dx} = e^{lnx} = x$$

$$y.(I.F) = \int Q(x).(I.F)dx + c$$

$$y.x = \int (x^2 - 1)xdx + c$$

$$y.x = \int (x^3 - x)dx + c \quad \rightarrow \quad yx = \frac{x^4}{4} - \frac{x^2}{2} + c$$

Example (2): Compute the general solution of differential equation $\frac{dr}{d\theta} + r\tan(\theta) = \cos(\theta)$

Solve:

$p(\theta) = \tan(\theta) \quad , \quad Q(\theta) = \cos(\theta)$

$I.F = e^{\int p(\theta)d\theta}$

$$I.F = e^{\int \tan(\theta)\, d\theta} = e^{\int \frac{\sin(\theta)}{\cos(\theta)} d\theta} = e^{-ln|\cos(\theta)|} = (\cos(\theta))^{-1} = \frac{1}{\cos(\theta)}$$

$$r.(I.F) = \int Q(\theta).(I.F)d\theta + c$$

$$r.\frac{1}{\cos(\theta)} = \int \cos(\theta)\frac{1}{\cos(\theta)}d\theta + c$$

$$r.\sec(\theta) = \int 1d\theta + c$$

$$r.\sec(\theta) = \theta + c$$

Example (3): Compute the general solution of differential equation

$$dy + 2xydx = xe^{-x^2}dx$$

Solve:

$dy + 2xydx = xe^{-x^2}dx \qquad \div dx$

$\frac{dy}{dx} + 2xy = xe^{-x^2}$

$p(x) = 2x \quad , \quad Q(x) = xe^{-x^2}$

$I.F = e^{\int p(x)dx} \quad \rightarrow \quad I.F = e^{\int 2xdx} = e^{x^2}$

$$y.(I.F) = \int Q(x).(I.F)dx + c$$

$$y.e^{x^2} = \int xe^{-x^2}.e^{x^2}dx + c$$

$$y.e^{x^2} = \int xdx + c$$

$$y.e^{x^2} = \frac{x^2}{2} + c$$

Example (4): Find the particular solution to the equation $\acute{y} = 2(2x - y)$ $with\ Initial\ condition\ \ y(0) = 1$

Solve:

$$\frac{dy}{dx} = 4x - 2y \quad \rightarrow \quad \frac{dy}{dx} + 2y = 4x$$

$$p(x) = 2 \quad , \quad Q(x) = 4x$$

$$I.F = e^{\int p(x)dx} \quad \rightarrow \quad I.F = e^{\int 2dx} = e^{2x}$$

$$y.(I.F) = \int Q(x).(I.F)dx + c$$

$$y.e^{2x} = \int 4x.e^{2x}dx + c \quad \rightarrow \quad y.e^{2x} = 4\int x.e^{2x}dx + c$$

$\int x.e^{2x}dx$ الحل بالتجزئة $u = x \quad \rightarrow \quad du = dx$

$$dv = e^{2x}dx \quad \rightarrow \quad v = \frac{1}{2}e^{2x}$$

$$\int x.e^{2x}dx = uv - \int vdu = x\frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x}dx$$

$$\int x.e^{2x}dx = \frac{x}{2}e^{2x} - \frac{1}{2}\frac{1}{2}e^{2x} = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x}$$

$$y.e^{2x} = 4\left(\frac{x}{2}e^{2x} - \frac{1}{4}e^{2x}\right) + c$$

$y.e^{2x} = 2xe^{2x} - e^{2x} + c$ الحل العام

$$1.e^{0} = 2(0)e^{0} - e^{0} + c \quad \rightarrow \quad c = 0$$

$y.e^{2x} = 2xe^{2x} - e^{2x}$ الحل الخاص

Example (5): Compute the general solution of differential equation $dy + \frac{3y}{x}dx = x^{-3}e^{x}dx$

Solve:

$$dy + \frac{3y}{x}dx = x^{-3}e^{x}dx \quad \div dx$$

$$\frac{dy}{dx} + \frac{3y}{x} = x^{-3}e^{x}$$

$$p(x) = \frac{3}{x} \quad , \quad Q(x) = x^{-3}e^{x}$$

$$I.F = e^{\int p(x)dx} \quad \rightarrow \quad I.F = e^{\int \frac{3}{x}dx} = e^{3lnx} = e^{lnx^{3}} = x^{3}$$

$$y.(I.F) = \int Q(x).(I.F)dx + c$$

$$y.x^3 = \int x^{-3}e^x x^3\, dx + c$$

$$yx^3 = \int e^x\, dx + c$$

$$yx^3 = e^x + c$$

Example (6): Compute the general solution of differential equation $(x-2)\frac{dy}{dx} = 2(x-2)^3 + y$

Solve:

$$(x-2)\frac{dy}{dx} = 2(x-2)^3 + y \qquad \div (x-2)$$

$$\frac{dy}{dx} = 2(x-2)^2 + \frac{y}{(x-2)}$$

$$\frac{dy}{dx} - \frac{y}{(x-2)} = 2(x-2)^2$$

$$p(x) = \frac{-1}{(x-2)} \qquad , \quad Q(x) = 2(x-2)^2$$

$$I.F = e^{-\int \frac{1}{(x-2)}dx} = e^{-\ln(x-2)} = e^{\ln(x-2)^{-1}} = (x-2)^{-1} = \frac{1}{(x-2)}$$

$$y.(I.F) = \int Q(x).(I.F)dx + c$$

$$y\frac{1}{(x-2)} = \int 2(x-2)^2\frac{1}{(x-2)}dx + c$$

$$\frac{y}{(x-2)} = \int 2(x-2)\, dx + c \qquad \rightarrow \quad \frac{y}{(x-2)} = \int (2x-4)\, dx + c$$

$$\frac{y}{(x-2)} = x^2 - 4x + c$$

Example (7): Compute the general solution of differential equation

$$xdy = sinx\, dx - ydx$$

Solve:

$$xdy = sinx\, dx - ydx \qquad \div dx$$

$$x\frac{dy}{dx} = sinx - y$$

$$x\frac{dy}{dx} + y = sinx \qquad \div x$$

$$\frac{dy}{dx} + \frac{y}{x} = \frac{sinx}{x}$$

$$p(x) = \frac{1}{x} \quad , \quad Q(x) = \frac{sinx}{x}$$

$$I.F = e^{\int p(x)dx} \quad \rightarrow \quad I.F = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$y.(I.F) = \int Q(x).(I.F)dx + c$$

$$y.x = \int \frac{sinx}{x}.xdx + c$$

$$y.x = \int sinx\, dx + c$$

$$y.x = -cosx + c$$

Example (8): Compute the general solution of differential equation $ye^y dx = (y^3 + 2xe^y)dy$

Solve:

$$ye^y dx = (y^3 + 2xe^y)dy \qquad \div dy$$

$$ye^y \frac{dx}{dy} = (y^3 + 2xe^y)$$

$$ye^y \frac{dx}{dy} - 2xe^y = y^3 \qquad \div ye^y$$

$$\frac{dx}{dy} - \frac{2}{y}x = y^2e^{-y}$$

$$p(y) = \frac{-2}{y} \quad , \quad Q(y) = y^2e^{-y}$$

$$I.F = e^{\int p(y)dy} \quad \rightarrow \quad I.F = e^{-\int \frac{2}{y}dy} = e^{-2\ln y} = e^{\ln y^{-2}} = y^{-2}$$

$$x.(I.F) = \int Q(y).(I.F)dy + c$$

$$x.y^{-2} = \int y^2e^{-y}.y^{-2}dy + c$$

$x.y^{-2} = \int e^{-y}\, dy + c$

$x.y^{-2} = -e^{-y} + c$

H.W: Compute the general solution of differential equation

1) $x\frac{dy}{dx} - 3y = x^2$

2) $\frac{dy}{dx} + \frac{y}{x} = 1$

3) $xdy + ydx = ydy$

4) $x\acute{y} + 3y = \frac{sinx}{x^2}$

Converting a nonlinear differential equation to a linear equation

تحويل المعادلة التفاضلية غير الخطية الى معادلة خطية

In some cases, the exponent of the dependent variable appears to be greater than one in these equations. The differential equation is converted to a linear equation by converting the dependent variable to another variable. An example of this type of equation is the Bernoulli formula, which takes the following form:

$$\frac{dy}{dx} + p(x)y = Q(x)y^n \qquad \ldots(1)$$

Where (n) is a positive integer >1.

في بعض الحالات يظهر اس المتغير المعتمد اكبر من الواحد في هذه المعادلات يتم تحويل المعادلة التفاضلية الى خطية عن طريق تحويل المتغير المعتمد الى متغير اخر مثالاً على هذا النوع من المعادلات صيغة برنولي والتي تأخذ الشكل الاتي:

The equation can be converted to a linear form according to the following steps:

1- We multiply the Eq.(1) by y^{-n}

$$y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = Q(x) \qquad \ldots(2)$$

2- Let $w = y^{1-n} \quad \rightarrow \quad dw = (1-n)y^{-n}dy$

$$\frac{dw}{(1-n)y^{-n}} = dy$$

Sub y^{1-n} & dy in Eq.(2)

$$y^{-n}\frac{\frac{dw}{(1-n)y^{-n}}}{dx} + p(x)w = Q(x)$$

$$\frac{dw}{(1-n)dx} + p(x)w = Q(x) \qquad \ldots(3)$$

3- The Eq. (3) is transformed into a linear differential equation in terms of the dependent variable (w) and the independent variable (x) and is solved according to the steps for solving a linear differential equation. After completing the solution, the values are returned (w) to (y).

Example (1): Compute the general solution of differential equation

$$\boldsymbol{xdy + ydx = xy^3dx}$$

Solve:

$$xdy + ydx = xy^3\mathrm{dx} \qquad \div \mathrm{dx}$$

$$x\frac{dy}{dx} + y = xy^3 \qquad \div x$$

$$\frac{dy}{dx} + \frac{y}{x} = y^3 \qquad * y^{-3}$$

$$\boldsymbol{y}^{-3}\frac{dy}{dx} + \frac{y^{-2}}{x} = 1 \qquad \ldots(1)$$

$$let\ w = y^{1-n} = y^{-2} \quad \rightarrow \quad dw = -2y^{-3}dy \quad \rightarrow \quad dy = \frac{-dw}{2y^{-3}}$$

Sub y^{-2} & dy in Eq.(1)

$$\boldsymbol{y}^{-3}\frac{\frac{-dw}{2y^{-3}}}{dx} + \frac{w}{x} = 1 \quad \rightarrow \quad \frac{-dw}{2dx} + \frac{w}{x} = 1 \qquad * -2$$

$$\frac{dw}{dx} - \frac{2w}{x} = -2$$

$$\boldsymbol{p(x) = \frac{-2}{x}} \quad , \quad \boldsymbol{Q(x)} = -2$$

$$\boldsymbol{I.F = e^{\int p(x)dx} \qquad I.F = e^{\int \frac{-2}{x}dx} = e^{-2\int \frac{1}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}}$$

$$w.(I.F) = \int Q(x).(I.F)dx + c$$

$$w.\mathbf{x}^{-2} = \int -2\mathbf{x}^{-2}dx + c \qquad w.\mathbf{x}^{-2} = -2\frac{\mathbf{x}^{-1}}{-1} + c$$

$$w.\mathbf{x}^{-2} = \frac{2}{x} + c$$

$$y^{-2}.\mathbf{x}^{-2} = \frac{2}{x} + c$$

Example (2): Compute the general solution of differential equation

$$\boldsymbol{xdy - (y + xy^3(1 + \ln x))dx = 0}$$

Solve:

$$xdy-\left(y+xy^3(1+lnx)\right)dx=0 \qquad \div dx$$

$$x\frac{dy}{dx}-y-xy^3(1+lnx)=0 \qquad \div x$$

$$\frac{dy}{dx}-\frac{y}{x}-y^3(1+lnx)=0$$

$$\frac{dy}{dx}-\frac{y}{x}=y^3(1+lnx) \qquad * y^{-3}$$

$$y^{-3}\frac{dy}{dx}-\frac{y^{-2}}{x}=(1+lnx) \qquad \dots(1)$$

$$let\ w=y^{1-n}=y^{-2} \quad\rightarrow\quad dw=-2y^{-3}dy \quad\rightarrow\quad dy=\frac{-dw}{2y^{-3}}$$

Sub y^{-2} & dy in Eq.(1)

$$y^{-3}\frac{\frac{-dw}{2y^{-3}}}{dx}-\frac{w}{x}=(1+lnx)$$

$$\frac{-dw}{2dx}-\frac{w}{x}=(1+lnx) \qquad *-2$$

$$\frac{dw}{dx}+\frac{2w}{x}=-2(1+lnx)$$

$$p(x)=\frac{2}{x} \quad , \quad Q(y)=-2(1+lnx)$$

$$I.F=e^{\int p(x)dx} \qquad I.F=e^{\int\frac{2}{x}dx}=e^{2\int\frac{1}{x}dx}=e^{2\ln x}=e^{\ln x^2}=x^2$$

$$w.(I.F)=\int Q(x).(I.F)dx+c$$

$$w.x^2=\int -2(1+lnx)x^2dx+c \quad\rightarrow\quad w.x^2=-2\left(\int(x^2+x^2lnxdx)\right)+c$$

$$w.x^2=-2\left(\int x^2dx+\int x^2lnxdx\right)+c$$

$$w.x^2=-2\left(\frac{x^3}{3}+\int x^2lnxdx\right)+c \qquad \dots.(2)$$

$$\int x^2lnxdx \qquad u=lnx \qquad du=\frac{1}{x} \qquad \int dv=\int x^2dx \qquad v=\frac{x^3}{3}$$

$$\int x^2lnxdx=\frac{x^3}{3}lnx-\int\frac{x^3}{3}\frac{1}{x}\,dx=\frac{x^3}{3}lnx-\int\frac{x^2}{3}dx$$

$$\int x^2lnxdx=\frac{x^3}{3}lnx-\frac{x^3}{9}$$

Sub $\int x^2 ln x dx$ in Eq.(2)

$$w.x^2 = -2\left(\frac{x^3}{3} + \frac{x^3}{3} ln x - \frac{x^3}{9}\right) + c$$

$$w.x^2 = \frac{-2}{3}x^3 - \frac{2}{3}x^3 ln x + \frac{2}{9}x^3 + c$$

$$w.x^2 = \frac{-4}{9}x^3 - \frac{2}{3}x^3 ln x + c$$

$$y^{-2}.x^2 = \frac{-4}{9}x^3 - \frac{2}{3}x^3 ln x + c$$

H.W: Compute the general solution of differential equation $\boldsymbol{\frac{dy}{dx} + y = y^2 cos x}$