Convex and Concave function

It has four cases:

1) The function $f(x_1, x_2, \cdots, x_n)$ is a convex function if and only if the matrix of second derivative or hessian matrix is positive, semi definite and the principal minor determinants of this matrix are all non−negative.
2) The function $f(x_1, x_2, \cdots, x_n)$ is a strictly convex if the hessian matrix is positive definite.
3) The function $f(x_1, x_2, \cdots, x_n)$ is a concave function if and only if the hessian matrix is negative semi definite.
4) The function $f(x_1, x_2, \cdots, x_n)$ is a strictly concave if the hessian matrix is negative definite.

Example: for each of the following function determine whether convex or concave or neither?

1) $f(x) = x_1x_2 - x_1^2 - x_2^2$
2) $f(x) = x_1^2 + 3x_1x_2 + 2x_2^2$
3) $f(x) = 3x_1 + 2x_1^2 + 4x_2 + x_2^2 - 2x_1x_2$
4) $f(x) = 2x_1^2 - 24x_1 + 2x_2^2 - 8x_2 + 2x_3^2 - 12x_3 + 200$
5) $f(x) = 3x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 + 2x_2x_3 - 6x_1 - 4x_2 - 2x_3$

Sol (1): $\boldsymbol{f(x) = x_1x_2 - x_1^2 - x_2^2}$

$$\frac{\partial f}{\partial x_1} = x_2 - 2x_1 \,, \quad \frac{\partial^2 f}{\partial x_1^2} = -2 \,, \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = 1$$

$$\frac{\partial f}{\partial x_2} = x_1 - 2x_2 \,, \quad \frac{\partial^2 f}{\partial x_2^2} = -2 \,, \quad \frac{\partial^2 f}{\partial x_2 \partial x_1} = 1$$

$$H(x) = \begin{bmatrix} \frac{\partial^2 f}{dx_1^2} & \frac{\partial^2 f}{dx_1 dx_2} \\ \frac{\partial^2 f}{dx_2 dx_1} & \frac{\partial^2 f}{dx_2^2} \end{bmatrix}, H = H^T$$

$$H = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$$

$$H_1 = [a_{11}] = [-2] = -2 < 0$$

$$H_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} = 4 - 1 = 3 > 0$$

Hence (H) is negative definite, so the function is strictly concave.

Sol (2): $\boldsymbol{f(x) = x_1^2 + 3x_1x_2 + 2x_2^2}$

$$\frac{\partial f}{\partial x_1} = 2x_1 + 3x_2 \,, \quad \frac{\partial^2 f}{\partial x_1^2} = 2 \,, \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = 3$$

$$\frac{\partial f}{\partial x_2} = 3x_1 + 4x_2 \ , \quad \frac{\partial^2 f}{\partial x_2^2} = 4 \ , \quad \frac{\partial^2 f}{\partial x_2 \partial x_1} = 3$$

$$H = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

$$H_1 = [a_{11}] = [2] = 2 > 0$$

$$H_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} = 8 - 9 = -1 < 0$$

Hence (H) is indefinite, so the function is neither convex nor concave.

Sol (3): $f(x) = 3x_1 + 2x_1^2 + 4x_2 + x_2^2 - 2x_1x_2$

$$\frac{\partial f}{\partial x_1} = 3 + 4x_1 - 2x_2 \ , \quad \frac{\partial^2 f}{\partial x_1^2} = 4 \ , \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = -2$$

$$\frac{\partial f}{\partial x_2} = 4 + 2x_2 - 2x_1 \ , \quad \frac{\partial^2 f}{\partial x_2^2} = 2 \ , \quad \frac{\partial^2 f}{\partial x_2 \partial x_1} = -2$$

$$H = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}$$

$$H_1 = [a_{11}] = [4] = 4 > 0$$

$$H_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} = 8 - 4 = 4 > 0$$

Hence (H) is positive definite, so the function is strictly convex.

Sol (4): $f(x) = 2x_1^2 - 24x_1 + 2x_2^2 - 8x_2 + 2x_3^2 - 12x_3 + 200$

$$\frac{\partial f}{\partial x_1} = 4x_1 - 24 \ , \quad \frac{\partial^2 f}{\partial x_1^2} = 4 \ , \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0 \ , \quad \frac{\partial^2 f}{\partial x_1 \partial x_3} = 0$$

$$\frac{\partial f}{\partial x_2} = 4x_2 - 8 \ , \quad \frac{\partial^2 f}{\partial x_2^2} = 4 \ , \quad \frac{\partial^2 f}{\partial x_2 \partial x_1} = 0 \ , \quad \frac{\partial^2 f}{\partial x_2 \partial x_3} = 0$$

$$\frac{\partial f}{\partial x_3} = 4x_3 - 12 \ , \quad \frac{\partial^2 f}{\partial x_3^2} = 4 \ , \quad \frac{\partial^2 f}{\partial x_3 \partial x_1} = 0 \ , \quad \frac{\partial^2 f}{\partial x_3 \partial x_2} = 0$$

$$H = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$H_1 = [a_{11}] = [4] = 4 > 0$$

$$H_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 16 - 0 = 16 > 0$$

$$H_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & 2_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 64 - 0 = 64 > 0$$

Hence (H) is positive definite so the function strictly convex.

Sol (5): $f(x) = 3x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3 - 6x_1 - 4x_2 - 2x_3$

$$\frac{\partial f}{\partial x_1} = 6x_1 - 2x_2 - 6 \ , \quad \frac{\partial^2 f}{\partial x_1^2} = 6 \ , \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = -2 \ , \quad \frac{\partial^2 f}{\partial x_1 \partial x_3} = 0$$

$$\frac{\partial f}{\partial x_2} = 4x_2 - 2x_1 + 2x_3 - 4 \ , \quad \frac{\partial^2 f}{\partial x_2^2} = 4 \ , \quad \frac{\partial^2 f}{\partial x_2 \partial x_1} = -2 \ , \quad \frac{\partial^2 f}{\partial x_2 \partial x_3} = 2$$

$$\frac{\partial f}{\partial x_3} = 2x_3 + 2x_2 - 2 \ , \quad \frac{\partial^2 f}{\partial x_3^2} = 2 \ , \quad \frac{\partial^2 f}{\partial x_3 \partial x_1} = 0 \ , \quad \frac{\partial^2 f}{\partial x_3 \partial x_2} = 2$$

$$H = \begin{bmatrix} 6 & -2 & 0 \\ -2 & 4 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

$$H_1 = [a_{11}] = [6] = 6 > 0$$

$$H_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} = 24 - 4 = 20 > 0$$

$$H_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & 2_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 6 & -2 & 0 \\ -2 & 4 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

$$= 6(8-4) - (-2)(-4) + 0(-4-4)$$

$$= 6(4) - 8$$

$$= 24 - 8 = 16 > 0$$

Hence (H) is positive definite so the function strictly convex.