Vectors and the Dot Product in Space

المتجهات والضرب النقطي في الفضاء

Definitions and Theorems: **تعاريف ونظريات**

- Vectors in space are denoted by $v = \langle v_1, v_2, v_3 \rangle$, where v_1, v_2 & v_3 are the components of the vector. The **<u>zero vector</u>** is $0 = \langle 0,0,0 \rangle$ and the **<u>standard unit vectors</u>** are $i = \langle 1,0,0 \rangle$, $j = \langle 0,1,0 \rangle$ and $k = \langle 0,0,1 \rangle$.

يشار إلى المتجهات في الفضاء بواسطة $v = \langle v_1, v_2, v_3 \rangle$، حيث ان v_1, v_2 & v_3 هي مكونات المتجه. المتجه الصفري هو $0 = \langle 0,0,0 \rangle$ ومتجهات الوحدة القياسية هي $i = \langle 1,0,0 \rangle$, $j = \langle 0,1,0 \rangle$ and $k = \langle 0,0,1 \rangle$.

- The **<u>length</u>** of the vector v is $\|v\| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2}$

- Two vectors are **<u>equal</u>** if they have the same components that is, if they have the same length and direction.

يكون المتجهان متساويين إذا كان لهما نفس المكونات، أي إذا كان لهما نفس الطول والاتجاه.

- The **<u>dot product</u>** of $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$ is

$$u * v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Properties the dot product:

1- u*v = v*u Exchange process
2- v*v=$\|v\|^2$
3- u*v =0 Two vectors are **<u>orthogonal</u>**

- If θ is the angle between the two nonzero vectors u and v, then

$$cos\theta = \frac{u * v}{\|u\|\|v\|}$$

إذا كانت θ هي الزاوية بين المتجهين غير الصفريين u و v، إذن

Example (1): Find the length of the vectors $v = \langle 1,3,4 \rangle$

Solve:

$$\|v\| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2} = \sqrt{1^2 + 3^2 + 4^2} = \sqrt{1 + 9 + 16} = \sqrt{26} = 5.1$$

Example (2): Find the dot product of the vectors $u = \langle 2,4,-3 \rangle$ and $v = \langle 5,0,4 \rangle$

Solve:

$$u * v = u_1 v_1 + u_2 v_2 + u_3 v_3$$
$$u * v = (2 * 5) + (4 * 0) + (-3 * 4) = 10 + 0 - 12 = -2$$

Example (3): Find the dot product of the vectors $u = \langle 2,-1,1\rangle$ and $v = \langle 1,0,-1\rangle$

Solve:

$$u * v = u_1v_1 + u_2v_2 + u_3v_3$$
$$u * v = (2 * 1) + (-1 * 0) + (1 * -1) = 2 + 0 - 1 = 1$$

Example (4): Find the angle between the vectors u = 3i + 4j and v = 2j+3k.

Solve:

$u = \langle 3,4,0\rangle$ and $v = \langle 0,2,3\rangle$

$$cos\theta = \frac{u * v}{\|u\|\|v\|}$$
$$u * v = u_1v_1 + u_2v_2 + u_3v_3$$
$$u * v = (3 * 0) + (4 * 2) + (0 * 3) = 0 + 8 + 0 = 8$$
$$\|u\| = \sqrt{{u_1}^2 + {u_2}^2 + {u_3}^2} = \sqrt{3^2 + 4^2 + 0^2} == \sqrt{9 + 16} = \sqrt{25} = 5$$
$$\|v\| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2} = \sqrt{0^2 + 2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13} = 3.61$$
$$cos\theta = \frac{8}{(5)(3.61)} = \frac{8}{18.05} = 0.44$$

Example (5): Find the angle between the vectors u = 2i + 3k and v = -i+2j+4k.

Solve:

$u = \langle 2,0,3\rangle$ and $v = \langle -1,2,4\rangle$

$$cos\theta = \frac{u * v}{\|u\|\|v\|}$$
$$u * v = u_1v_1 + u_2v_2 + u_3v_3$$
$$u * v = (2 * -1) + (0 * 2) + (3 * 4) = (-2) + 0 + 12 = 10$$
$$\|u\| = \sqrt{{u_1}^2 + {u_2}^2 + {u_3}^2} = \sqrt{2^2 + 0^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13} = 3.61$$
$$\|v\| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2} = \sqrt{(-1)^2 + 2^2 + 4^2} = \sqrt{1 + 4 + 16} = \sqrt{21} = 4.58$$
$$cos\theta = \frac{10}{(3.61)(4.58)} = \frac{10}{16.53} = 0.60$$

Example (6): Find the orthogonal vectors of the vectors $u = \langle 3,-1,2\rangle$, $v = \langle -4,0,2\rangle$ and $w = \langle 1,-1,-2\rangle$.

Solve:

$$u * v = u_1v_1 + u_2v_2 + u_3v_3$$
$$u * v = (3 * -4) + (-1 * 0) + (2 * 2) = (-12) + 0 + 4 = -8$$
$$u * w = u_1w_1 + u_2w_2 + u_3w_3$$
$$u * w = (3 * 1) + (-1 * -1) + (2 * -2) = 3 + 1 - 4 = 0$$

The vectors u and w are orthogonal

$$v * w = v_1w_1 + v_2w_2 + v_3w_3$$

$$v * w = (-4 * 1) + (0 * -1) + (2 * -2) = -4 + 0 - 4 = -8$$

Example (7): let $\|v\| = 5\,, \|u\| = 8\,, \theta = \frac{2\pi}{3}$ find $\|2v - 3u\|$

Solve:

$$\|2v - 3u\|^2 = 4(v * v) - 12(v * u) + 9(u * u)$$

$$= 4\|v\|^2 - 12\|v\|\|u\| cos\frac{2\pi}{3} + 9\|u\|^2$$

$$= 4(5)^2 - 12(5)(8)(-0.5) + 9(8)^2 = 100 + 240 + 576 = 916$$

$$\|2v - 3u\| = \sqrt{916} = 30.27$$