Duality and Duality Theory

To every linear program there is a dual linear program with which it is intimately connected. We first state this duality for the standard programs. Suppose we have a Lpp :

$$\begin{aligned} &Min\ Z = c^T x \\ &S.t.: \quad Ax \ge b \\ &\qquad\quad x \ge 0 \end{aligned} \qquad \text{..........}(*)$$

Then the Lpp :

$$\begin{aligned} &Max\ V = b^T y \\ &S.t.: \quad A^T y \le c \\ &\qquad\quad y \ge 0 \end{aligned} \qquad \text{..........}(**)$$

is called the dual of the Lpp $(*)$, $(*)$is called primal Lpp.

Therefore, the dual of the standard minimum problem $(**)$ is the standard maximum problem $(*)$.

Note :

If a standard problem and its dual are both feasible, then both are bounded feasible.

If there exists feasible x and y for a standard maximum problem (1) and its dual (2) such that $c_j x_j = b_i y_i$, then both are optimal for their respective problems.

Every Lpp has its dual.

Example : Consider a Lpp :

$$\begin{aligned} &Min\ Z = x_1 + 2x_2 - x_3 \\ &S.t.: \quad x_1 + x_2 - x_3 \le 1 \\ &\qquad 2x_1 - x_2 + x_3 \le 4 \\ &\qquad x_1 + x_2 + x_3 \le 2 \\ &\qquad x_1 - 2x_2 + x_3 \le 3 \\ &\qquad x_1 , x_2 , x_3 \ge 0 \end{aligned} \qquad \text{..........}(A)$$

Find its dual Lpp.

$$
\begin{aligned}
&Max\ \ V = y_1 + 4y_2 + 2y_3 + 4y_4 \\
&S.t.: \quad y_1 + 2y_2 + y_3 + y_4 \geq 1 \\
&\qquad y_1 - y_2 + 2y_3 - 2y_4 \geq 2 \\
&\qquad - y_1 + y_2 + y_3 + y_4 \geq -1 \\
&\qquad y_1\ ,\ \ y_2\ \ ,\ y_3\ ,\ \ y_4 \geq 0
\end{aligned}
\qquad \qquad \text{..........}(B)
$$

Some problems

***** If the original problem in $(*)$ is :

$$Max\ Z\ = \sum_{j=1}^{n} c_j\ x_j$$

Then we reversed it as :

$$Min\ \ V\ = -\ \sum_{j=1}^{n} c_j\ x_j$$

****** If in the set of the constraints in $(*)$ we have some constants of the type

$$\sum_{j=1}^{n} a_{i\,j} x_j\ \leq b_i\ \ ,\ i = 1\ ,\ 2\ ,\ 3\ ,\ .\ .\ .\ .\ ,\ m$$

So we well reversed it by multiplying by (-1) as :

$$\sum_{j=1}^{n} -\ a_{i\,j} x_j\ \geq\ -\ b_i\ \ ,\ i = 1\ ,\ 2\ ,\ 3\ ,\ .\ .\ .\ .\ ,\ m$$

******* Also if we have some constraints of the type in $(*)$:

$$\sum_{j=1}^{n} a_{i\,j} x_j\ = b_i\ \ ,\ i = 1\ ,\ 2\ ,\ 3\ ,\ .\ .\ .\ .\ ,\ m$$

Then we will write it as :

$$\sum_{j=1}^{n} a_{i\,j} x_j\ \leq b_i\ \ ,\ i = 1\ ,\ 2\ ,\ 3\ ,\ .\ .\ .\ .\ ,\ m$$

$$\sum_{j=1}^{n} a_{i\,j} x_j\ \geq b_i\ \ ,\ i = 1\ ,\ 2\ ,\ 3\ ,\ .\ .\ .\ .\ ,\ m$$

Example :

Consider a Lpp :

$$
\begin{aligned}
&Min\ Z = x_1 + 2x_2 \\
&S.t.: -x_1 + x_2 \le 10 \\
&\qquad x_1 + x_2 \le 6 \\
&\qquad x_1 + x_2 = 2 \\
&\qquad x_1 + 3x_2 \ge 6 \\
&\qquad x_1 \ , \ x_2 \ge 0
\end{aligned}
\qquad \text{..........} (*)
$$

Solution :

$$
\begin{aligned}
&Min\ Z = x_1 + 2x_2 \\
&S.t.: \quad x_1 - x_2 \ge -10 \\
&\qquad - x_1 - x_2 \ge -6 \\
&\qquad x_1 + x_2 \ge 2 \\
&\qquad -x_1 - x_2 \ge -2 \\
&\qquad x_1 + 3x_2 \ge 6 \\
&\qquad x_1 \ , \ x_2 \ge 0
\end{aligned}
\qquad \text{..........} (**)
$$

So the dual of this problem is :

$$
\begin{aligned}
&Max\ V = -10y_1 - 6y_2 + 2y_3 - 2y_4 + 6y_5 \\
&S.t.: \quad y_1 - y_2 + y_3 - y_4 + y_5 \le -1 \\
&\qquad -3y_1 - y_2 + y_3 - y_4 + 3y_5 \le -2 \\
&\qquad y_1 \ , \ y_2 \ , y_3 \ , \ y_4 \ , \ y_5 \ge 0
\end{aligned}
\qquad \text{..........} (***)
$$