Lagranges multipliers :

The solution of a constrained optimization problem can often be found by using the Lagrange multipliers. In general, the Lagranges multipliers is the sum of the original objective function and a term that involves the functional constraint a Lagrange multipliers λ. We define the Lagrange multipliers as:

$$L(x_1,x_2,\dots,x_n,\lambda_1,\lambda_2,\dots,\lambda_m)=f(x_1,x_2,\dots,x_n)-\lambda_1 g_1(x_1,x_2,\dots,x_n)-\lambda_2 g_2(x_1,x_2,\dots,x_n)-\dots-\lambda_m g_m(x_1,x_2,\dots,x_n)\} \qquad (2)$$

The stationary values of (1) in this case are given by calculating the partial derivatives with respect to these variables, we obtain the first-order conditions of the optimization problem:

$$\left.\begin{array}{l} \dfrac{\partial h}{\partial x_1}=0,\dfrac{\partial h}{\partial x_2}=0,\dots,\dfrac{\partial h}{\partial x_n}=0 \longrightarrow (n)\ equations \\ \dfrac{\partial h}{\partial \lambda_1}=0,\dfrac{\partial h}{\partial \lambda_2}=0,\dots,\dfrac{\partial h}{\partial \lambda_m}=0 \longrightarrow (m)\ equations \end{array}\right\} \qquad (III)$$

The optimum points of function f are the solutions of $n+m$ equations. May be used to find the values of $x_1, x_2,\dots, x_n, \lambda_1, \lambda_2,\dots, \lambda_m$ are called Lagrange's multipliers.

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Example :

$$f = -x_1^2 - 2x_2 + x_2^2$$
$$S.t: \ -x_1^2 - x_2^2 + 1 \geq 0$$

Solution :

$$L(x_1, x_2, x_3, \lambda) = -x_1^2 - 2x_1 + x_2^2 - \lambda(-x_1^2 - x_2^2 + 1 + x_3^2)$$

$$L_{x_1} = -2x_1 - 2 + 2x_1\lambda = 0 \Rightarrow x_1(\lambda - 1) = 1$$

$$L_{x_2} = 2x_2 + 2x_2\lambda = 0 \Rightarrow x_2(\lambda + 1) = 0$$

$$L_{x_3} = -2x_3\lambda = 0 \Rightarrow x_3\lambda = 0$$

$$L_\lambda = +x_1^2 + x_2^2 - 1 - x_3^2 = 0 \Rightarrow +x_1^2 + x_2^2 - x_3^2 = +1$$

Point (A): $\lambda = -1$; $x_1 = -\frac{1}{2}$; $x_2 = \mp\frac{\sqrt{3}}{2}$; $x_3 = 0$

$\left(-\frac{1}{2}\right)(-1-1) = 1$ $\left(-\frac{1}{2}\right)(-1-1) = 1$

$\left(\frac{\sqrt{3}}{2}\right)(-1+1) = 0$ $\left(-\frac{\sqrt{3}}{2}\right)(-1+1) = 0$

$(0)(-1) = 0$ $(0)(-1) = 0$

$\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + (0)^2 = 1$ $\left(-\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2 + (0)^2 = 1$

$$-\frac{1}{4} + 1 = x_2^2 \Rightarrow x_2^2 = \frac{3}{4} \Rightarrow x_2 = \mp\frac{\sqrt{3}}{2}$$

Point (B): $\lambda = 2$; $x_2 = 0$; $x_3 = 0$; $x_1^2 = 1$

$x_1^2 = 1 \Rightarrow x_1 = \mp 1 \Rightarrow x_1 = 1$

$$(1)\,(2-1) = 1$$
$$(0)\,(2+1) = 0$$
$$(0)\,(2) = 0$$
$$(1)^2 + (0)^2 + (0)^2 = 1$$

Point (C): $\lambda = 0$; $x_1 = -1$; $x_2 = 0$; $x_3 = 0$

$$(-1)\ (0-1)=1$$
$$(0)(0+1)=0$$
$$(0)(0)=0$$
$$(-1)^2+(0)^2+(0)^2=1$$

Example :

Use Lagrange multipliers method to solve:

$$f \ = x_1^2 + x_2^2$$
$$S.t: \ 4-x_1 - x_2 = 0$$

Solution :

$$L(x_1, x_2, \lambda) = -x_1^2 + x_2^2 + \lambda(4 - x_1 - x_2)$$

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda = 0 \qquad ....(1)$$
$$\frac{\partial L}{\partial x_2} = 2x_2 - \lambda = 0 \qquad ....(2)$$
$$\frac{\partial L}{\partial \lambda} = 4 - x_1 - x_2 = 0 \qquad ....(3)$$

The solution to these equation is : $x_1 = x_2 = 2$ and $\lambda = 4$