Example (6): **Find the general and Special solution for the differential equation** $\frac{d^4y}{dx^4} = 5x^2$. **The initial condition** $y'''(0) = 2$, $y''(1) = -5$, $y'(0)=3$, $y(1) = 0$

Solve:

$\int \frac{d^4y}{dx^4} = \int 5x^2 dx \rightarrow \frac{d^3y}{dx^3} = \frac{5x^3}{3} + c_1 \rightarrow y''' = \frac{5x^3}{3} + c_1$

$\because\ y'''(0) = 2 \rightarrow 2 = 0 + c_1 \rightarrow c_1 = 2$

$$\int \frac{d^3y}{dx^3} = \int \frac{5x^3}{3} dx + \int 2\, dx$$

$$\frac{d^2y}{dx^2} = \frac{5x^4}{12} + 2x + c_2$$

$\because\ y''(1) = -5 \rightarrow -5 = \frac{5}{12} + 2 + c_2$

$$-7 - \frac{5}{12} = c_2 \rightarrow \frac{-89}{12} = c_2$$

$$\int \frac{d^2y}{dx^2} = \int \frac{5x^4}{12} dx + \int 2x\, dx - \int \frac{89}{12} dx$$

$$\frac{dy}{dx} = \frac{x^5}{12} + x^2 - \frac{89}{12}x + c_3$$

$\because\ y'(0) = 3 \rightarrow 3 = 0 + 0 - 0 + c_3 \rightarrow c_3 = 3$

$$\int \frac{dy}{dx} = \int \frac{x^5}{12} dx + \int x^2\, dx - \int \frac{89}{12} x dx + \int 3dx$$

$$y = \frac{x^6}{72} + \frac{x^3}{3} - \frac{89x^2}{24} + 3x + c_4$$

$\because\ y(1) = 0 \rightarrow 0= \frac{1}{72} + \frac{1}{3} - \frac{89}{24} + 3 + c_4$

$$0 = \frac{(1 + 24 - 267 + 216)}{72} + c_4 \rightarrow c_4 = \frac{26}{72}$$

$$y = \frac{x^6}{72} + \frac{x^3}{3} - \frac{89x^2}{24} + 3x + \frac{26}{72}$$

Example (7): What value of c makes $y = e^{cx}$ a solution to the differential equation $y'' + 5y' + 6y = 0$

Solve:

$y' = ce^{cx} \rightarrow y'' = c^2e^{cx}$

$y'' + 5y' + 6y = 0 \rightarrow c^2e^{cx} + 5ce^{cx} + 6e^{cx} = 0$

$\boldsymbol{e^{cx}}(\boldsymbol{c^2} + 5\boldsymbol{c} + 6) = 0$

$\boldsymbol{e^{cx}}$ ***is impossible* 0**

$\therefore (\boldsymbol{c^2} + 5\boldsymbol{c} + 6) = 0 \;\; \rightarrow (c+3)(c+2) = 0$

C=-3 or c=-2

1. 6 Formation of the differential equation from the general solution

To form the differential equation, the dependent variable is differentiated with respect to the independent variable a number of times equal to the number of constants in the general solution.

تكوين المعادلة التفاضلية من الحل العام

لتكوين المعادلة التفاضلية يتم مفاضلة المتغير المعتمد بالنسبة للمتغير المستقل لعدد من المرات مساوياً لعدد الثوابت في الحل العام.

Example (1): Find the differential equation whose solution is general $y = A\, e^{x} - x$ where A is an arbitrary constant.

Solve:

$y = A\, e^{x} - x$

$\acute{y} = A\, e^{x} - 1 \qquad \rightarrow \quad A\, e^{x} = \acute{y} + 1$

$A = e^{-x}(\acute{y} + 1)$

Sub A in the general solution $y = A\, e^{x} - x$

$y = e^{x} e^{-x}(\acute{y} + 1) - x$

$y - \acute{y} - 1 + x = 0$

Example (2): Find the differential equation whose solution is general $y = A\, x^2 + A^2$ where A is an arbitrary constant.

Solve:

$y = A\, x^2 + A^2 \qquad \rightarrow \acute{y} = 2A\, x$

$$A = \frac{\acute{y}}{2x}$$

Sub A in the general solution $y = A\, x^2 + A^2$

$$y = \left(\frac{\acute{y}}{2x}\right) x^2 + \left(\frac{\acute{y}}{2x}\right)^2$$

$$y - \frac{1}{2}\acute{y}\, x - \frac{1}{4x^2}(\acute{y})^2 = 0$$

Example (3): Find the differential equation whose solution is general $y = A x + Bx^4$ where A & B are an arbitrary constants.

Solve:

$y = A x + Bx^4$

$\acute{y} = A + 4B\,x^3 \qquad \dots.(1)$

$\acute{\acute{y}} = 12B\,x^2 \qquad \dots.(2)$

$$B = \frac{\acute{\acute{y}}}{12\,x^2}$$

Sub B in above Eq(1):

$$\acute{y} = A + 4\frac{\acute{\acute{y}}}{12\,x^2}\,x^3$$

$$\acute{y} = A + \frac{\acute{\acute{y}}x}{3} \quad \rightarrow \quad A = \acute{y} - \frac{\acute{\acute{y}}x}{3}$$

Sub A & B in the general solution $y = A x + Bx^4$

$$y = (\acute{y} - \frac{\acute{\acute{y}}x}{3})\,x + \frac{\acute{\acute{y}}}{12\,x^2}x^4$$

$$y = \acute{y}x - \frac{\acute{\acute{y}}x^2}{3} + \frac{\acute{\acute{y}}x^2}{12}$$

$$y = \acute{y}x - \frac{3\acute{\acute{y}}x^2}{12}$$

$$y - \acute{y}x + \frac{1}{4}\acute{\acute{y}}x^2 = 0$$

Example (4): Find the differential equation whose solution is general $y = A\,e^{x+B}$ where A & B are an arbitrary constant.

Solve:

$y = A\,e^{x+B}$

$\acute{y} = A\,e^{x+B} \qquad \dots.(1)$

$\acute{\acute{y}} = A\,e^{x+B} \qquad \dots.(2)$

$$\acute{\acute{y}} = y \quad \rightarrow \quad \acute{\acute{y}} - y = 0$$

Example (5): Find the differential equation whose solution is general $y = Ax^2 + Bx + 1$ where A & B are an arbitrary constant.

Solve:

$y = Ax^2 + Bx + 1$

$\acute{y} = 2Ax + B \quad \ldots\ldots(1)$

$\acute{\acute{y}} = 2A \quad \ldots.(2)$

$A = \frac{\acute{\acute{y}}}{2}$

Sub A in above Eq(1):

$\acute{y} = 2\left(\frac{\acute{\acute{y}}}{2}\right)x + B \quad \rightarrow B = \acute{y} - \acute{\acute{y}}x$

Sub A & B in the general solution $y = Ax^2 + Bx + 1$

$y = \frac{1}{2}\acute{\acute{y}}\,x^2 + (\acute{y} - \acute{\acute{y}}x)x + 1$

$y = \frac{1}{2}\acute{\acute{y}}\,x^2 + \acute{y}x - \acute{\acute{y}}x^2 + 1$

$y - \acute{y}x + \frac{1}{2}\acute{\acute{y}}\,x^2 - 1 = 0$

Example (6): Find the differential equation whose solution is general $y = e^{2x}(c_1 + c_2x)$ where c_1 & c_2 are an arbitrary constant.

Solve:

$y = e^{2x}(c_1 + c_2x) \quad \ldots\ldots(\mathbf{1})$

$\acute{y} = e^{2x}c_2 + 2(c_1 + c_2x)e^{2x} \quad \ldots.(2)$

$\acute{y} = e^{2x}c_2 + 2y \quad \ldots.(2)$

$\acute{\acute{y}} = 2e^{2x}c_2 + 2\acute{y} \quad \ldots.(3)$

Multi. Eq(2) by (-2):

$-2\acute{y} = -2e^{2x}c_2 - 4y \quad \ldots.(2)$

$\underline{\acute{\acute{y}} = 2e^{2x}c_2 + 2\acute{y} \quad \ldots.(3)}$ بالجمع

$\acute{\acute{y}} - 2\acute{y} = 2\acute{y} - 4y$

$\acute{\acute{y}} - 2\acute{y} - 2\acute{y} + 4y = 0 \quad \rightarrow \acute{\acute{y}} - 4\acute{y} + 4y = 0$

Example (7): Find the differential equation whose solution is general $y = e^{2x}(c_1\cos(x) + c_2\sin(x))$ where c_1 & c_2 are an arbitrary constant.

Solve:

$y = e^{2x}(c_1\cos(x) + c_2\sin(x)) \quad \ldots..(\mathbf{1})$

$\acute{y} = e^{2x}(-c_1\sin(x) + c_2\cos(x)) + 2(c_1\cos(x) + c_2\sin(x))e^{2x}$

$\acute{y} = e^{2x}(-c_1\sin(x) + c_2\cos(x)) + 2y \quad \ldots.(2)$

$\acute{\acute{y}} = e^{2x}(-c_1\cos(x) - c_2\sin(x)) + 2(-c_1\sin(x) + c_2\cos(x))e^{2x} + 2\acute{y} \quad \ldots.(3)$

$\acute{\acute{y}} = -y + 2(\acute{y} - 2y) + 2\acute{y} \quad \ldots.(3)$

$\acute{\acute{y}} = -y + 2\acute{y} - 4y + 2\acute{y}$

$\acute{\acute{y}} + 5y - 4\acute{y} = 0$

Because $\acute{y} - 2y = 2e^{2x}(-c_1\sin(x) + c_2\cos(x)) \quad \ldots.(2)$

Example (8): Find the differential equation whose solution is general $y^2 = 4Ax$ where A is an arbitrary constant.

Solve:

$y = \sqrt{4Ax}$

$y = 2\sqrt{A}(x)^{\frac{1}{2}} \quad \ldots(1)$

$\acute{y} = 2\,\frac{1}{2}\sqrt{A}(x)^{-\frac{1}{2}}$

$\acute{y} = \frac{1}{(x)^{\frac{1}{2}}}\sqrt{A} \quad \ldots(2)$

$\sqrt{A} = \acute{y}\,(x)^{\frac{1}{2}}$

Sub. $\sqrt{A}$ in Eq.(1)

$y = 2\,\acute{y}\,(x)^{\frac{1}{2}}(x)^{\frac{1}{2}} \quad \rightarrow y = 2\,\acute{y}x$

$y - 2\acute{y}x = 0$

H.W(1): Find the differential equation whose solution is general $y = A\sin x + B\cos x$ where A & B are an arbitrary constant.

H.W(2): Find the differential equation whose solution is general $y = c_1 e^{2x} + c_2 e^{3x}$ where c_1 & c_2 are an arbitrary constant**.**

H.W(3): Prove that $y = 2e^{x}$ is a general solution for the differential equation $(x-1)\acute{\acute{y}} + x\acute{y} - y = 0$.

H.W(4): Prove that $y = x \ln x - x$ is a general solution for the differential equation $x\acute{y} = x + y \quad , \; x > 0.$