Iterated integrals , Double Integrals and Volume
التكاملات التكرارية، التكاملات المزدوجة والحجم

Definition and Theorems:

- Iterated integrals are of the form:

$$\int_{y_1}^{y_2}\int_{x_1}^{x2} f(x,y)dxdy \quad or \quad \int_{x_1}^{x_2}\int_{y_1}^{y2} f(x,y)dydx$$

- The volume of a solid can be represented by a double integral. The properties of double integrals are:
  1) $\iint cf(x,y)dA = c\iint f(x,y)dA$
  2) $\iint [f(x,y)+g(x,y)]dA = \iint f(x,y)dA + \iint g(x,y)dA$

- Let f be integrable over the plane region R of area A. The average value of f over R is:

$$\frac{1}{A}\iint f(x,y)dA$$

Example (1): Calculate the iterated integrals $\int_2^4[\int_1^x 2xy\,dy]dx$

Solve:

$$\int_2^4[\int_1^x 2xy\,dy]dx = \int_2^4[xy^2]_1^x\,dx = \int_2^4[x(x^2)-x(1)]\,dx = \int_2^4[x^3-x]\,dx$$

$$\int_2^4[x^3-x]\,dx = [\frac{x^4}{4}-\frac{x^2}{2}]_2^4 = \left[\frac{4^4}{4}-\frac{4^2}{2}\right] - \left[\frac{2^4}{4}-\frac{2^2}{2}\right] = (64-8)-(4-2) = 54$$

Example (2): Calculate the iterated integrals $\int_0^{\frac{\pi}{2}}[\int_0^1 y\cos x\,dy]dx$

Solve:

$$\int_0^{\frac{\pi}{2}}[\int_0^1 y\cos x\,dy]dx = \int_0^{\frac{\pi}{2}}[\frac{y^2}{2}\cos x]_0^1\,dx = \int_0^{\frac{\pi}{2}}[\frac{1^2}{2}\cos x - \frac{0^2}{2}\cos x]\,dx = \int_0^{\frac{\pi}{2}}\frac{1}{2}\cos x\,dx$$

$$\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos x\,dx = \frac{1}{2}[\sin x]_0^{\frac{\pi}{2}} = \frac{1}{2}\left[\sin\frac{\pi}{2}-\sin 0\right] = \frac{1}{2}\,[1-0] = \frac{1}{2} = 0.5$$

Example (3): Calculate the volume below the surface $Z = 6-2y$ and above the rectangle given by $0 \le x \le 4\,,\ 0 \le y \le 2$.

Solve:

The volume is given by the double integral $\iint f(x,y)dA = \int_0^4[\int_0^2(6-2y)\,dy]dx$

$$\int_0^4 [\int_0^2 (6-2y)\, dy] dx = \int_0^4 [6y - y^2]_0^2\, dx = \int_0^4 [(6(2)-2^2) - (6(0)-0^2)]\, dx = \int_0^4 8\, dx$$

$$\int_0^4 8\, dx = [8x]_0^4 = 8(4) - 8(0) = 32$$

Example (4): Find the average value of $f(x,y) = \frac{1}{2}xy$ over the rectangular region R with vertices (0,0), (4,0), (4,3) and (0,3).

Solve:

The area of the region is $4 \times 3 = 12$ the average value is:

$$\frac{1}{A}\iint f(x,y) dA = \frac{1}{12}\int_0^4 [\int_0^3 \frac{1}{2}xy\, dy] dx = \frac{1}{12}\int_0^4 [\frac{1}{4}xy^2]_0^3\, dx$$

$$= \frac{1}{12}\int_0^4 [\frac{x(3^2)}{4} - \frac{x(0^2)}{4}]\, dx = \frac{9}{48}\int_0^4 x\, dx = \frac{3}{16}[\frac{x^2}{2}]_0^4 = \frac{3}{16}\left[\frac{4^2}{2} - \frac{0^2}{2}\right] = \frac{3}{2}$$

Example (5): Find the average value of $f(x,y) = x$ over the rectangular region R with vertices (0,0), (4,0), (4,2) and (0,2).

Solve:

The area of the region is $4 \times 2 = 8$ the average value is:

$$\frac{1}{A}\iint f(x,y) dA = \frac{1}{8}\int_0^4 [\int_0^2 [x\, dy] dx = \frac{1}{8}\int_0^4 [xy]_0^2\, dx = \frac{1}{8}\int_0^4 [x(2) - x(0)]\, dx$$

$$= \frac{2}{8}\int_0^4 x\, dx = \frac{1}{4}[\frac{x^2}{2}]_0^4 = \frac{1}{4}\left[\frac{4^2}{2} - \frac{0^2}{2}\right] = 2$$