Unconstrained Optimization

The field of optimization is concerned with the study of *maximization* and *minimization* of *mathematical* functions. Optimization means finding that value of x which maximizes or minimizes a given function $f(x)$.

One must choose between algorithms which use derivatives and those which do not. In general, methods which use derivatives are more powerful.

Many algorithms for minimizing $f(x)$ are in fact derived from algorithms for solving $f^{/}(x) = 0$.

An extremum (maximum or minimum) of $f(x)$ can be either global or local .

Definition

Given a function $f(x): R^n \to R$, we want to find a minimum of $f(x)$, i.e., to solve the following unconstrained optimization problem

$$\text{find } x \text{ to minimize } f(x).$$

x^* is the **global minimum** of $f(x)$, if :

$$f(x^*) \le f(x), \text{ for all } x \in R^n.$$

A point x^* is a **local minimum** of $f(x)$, if there exists $\varepsilon \succ 0$ such that :

$$f(x^*) \le f(x), \text{ for all } x, \ \|x - x^*\| < \varepsilon$$

1. Unconstrained optimization in one dimension

The simple unconstrained optimization problem in one dimension is of the form :

$$Minimize\ f(x), \quad \text{i.e.,} \quad \underset{x}{Min}\ f(x)$$

For a function $f : R \to R$.

In general there is no algorithm to find the global minimum of the function, since most algorithms are derived by using the local approximations of the function. What can be achieved is to find a local minimum numerically. There is no formula that can find a local minimum of an arbitrary function exactly.

Let us first review the Taylor expansion which is very frequently used in this subject, also for this course.

Taylor's Theorem in one dimension

Let $f : R \to R$ be a smooth function. Given $h \in R,$ the expansion of $f\ (x+h)$ at the point x is :

$$f\ (x+h) = f(x) + hf^{/}(x) + \frac{h^2}{2} f^{//}(x) + \ldots\ldots + \frac{h^n}{n!} f^{n}(x)$$

which is called the Taylor series.

The following theorems provide some necessary and sufficient conditions for local minimum.

Theorem (A necessary condition)

Let $f(x)$ be continuously differentiable. If x^* is a local minimum of $f(x),$ then $f(x^*) = 0$.

We call x a Stationary (critical) point of $f(x),$ if $f^{/}(x^*) = 0$.

Theorem (A sufficient condition)

Let $f(x)$ be continuously differentiable. If there exists a point x^*, where $f^{/}(x^*) = 0$, and $f^{//}(x^*) > 0$, then $f(x)$ achieves a **local minimum** at x^*.

A similar results says that if $f^{/}(x^*) = 0$, and $f^{//}(x^*) < 0$, then $f(x)$ achieves a **local maximum** at x^*.

Theorem (A sufficient condition)

Let $f^{/}(x^*) = f^{//}(x^*) = \ldots\ldots = f^{n-1}(x^*) = 0$ but $f^{n}(x^*) \neq 0$. Then $f(x^*)$ is **(i)** a minimum value of $f(x)$ if $f^{n}(x^*) > 0$ and n is even; (ii) a maximum value of $f(x)$ if $f^{n}(x^*) < 0$ and n is even; (iii) neither a maximum nor a minimum if n is odd.

Example :

Consider the function $f(x) = x^3 + 3x^2 - 24x + 17$.

Solution :

Differentiae and set to Zero :

$$f^{/}(x^*) = 3x^2 + 6x - 24 = 0$$

Solve the quadratic equation :

$$x^2 + 2x - 8 = 0$$

For the critical point :

$$x = \frac{2 \overset{+}{-} \sqrt{4+32}}{2} = 2 \ , -4$$

The second differentiae :

$$f^{//}(x^*) = 6x + 6$$

At :

$$f^{//}(2) = 18 \quad > 0 \quad \Rightarrow \min imum$$
$$f^{//}(-4) = -18 \quad < 0 \quad \Rightarrow \max imum$$

Example :

Consider the function $f(x) = x^3$.

Solution :

Differentiae and set to Zero :

$$f^{/}(x^*) = 3x^2 = 0$$

Stationary point at :

$$x = 0$$

The second derivative is :

$$f^{//}(x^*) = 6x$$

At $x = 0$, $f^{//}(0) = 0$ and hence we must investigate the next derivative :

$$f^{///}(x^*) = 6$$

Since $f^{///}(0) \neq 0$ at $x = 0$ is neither maximum nor minimum point, and it is an inflection point.