Limits and Continuity

الغايات والاستمرارية

Definitions and Theorems: **تعاريف ونظريات**

- Let f be a function of two variables defined, except possibly at (x_0, y_0) on an open disk centered at (x_0, y_0) and let L be a real number, then:

افترض أن f تكون دالة لمتغيرين محددين، باستثناء (x_0, y_0)، على قرص مفتوح متمركز في (x_0, y_0) وليكن L رقمًا حقيقيًا، فأن:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

If it is ($\lim f(x,y) = \frac{0}{0}$) then there are three ways to solve it:

1- factorization or multiplication by the conjugate التحليل أو الضرب بالمرافق

2- path method: طريقة المسار

If the limit contains two different paths and each path gives two different answers, then the limit does not exist (D.N.E).

اذا احتوت الغاية على مسارين مختلفين وكل مسار اعطى اجابتين مختلفين في هذه الحالة الغاية غير موجودة.

تستخدم هذه الطريقة عندما تكون قوة البسط اقل او تساوي درجة المقام.

3- polar method الطريقة القطبية

تستخدم هذه الطريقة عندما تفشل طريقة المسار (path method) وذلك عندما تكون قوة البسط اكبر من قوة المقام. هذه الطريقة هي عملية استبدال المتغيرات كالاتي:

$$y = r\sin\theta$$
$$x = r\cos\theta$$
$$r^2 = x^2 + y^2$$

ملاحظة: النتيجة التي يتم الحصول عليها عند استخدام طريقة **(path method)** هي اما الغاية غير موجودة **(D.N.E)** او فشل **(fails)** اما عند استخدام طريقة **(polar method)** يتم الحصول على الغاية موجودة ومساوية للصفر **(exist=0)** او الغاية غير موجودة **(D.N.E)**.

- A function of two variables is **continuous** at a point (x_0, y_0) if $f(x_0, y_0)$ is defined and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$$

Example (1): Calculate the limit $\lim_{(x,y)\to(1,2)} \frac{5x^2y}{x^2+y^2}$

Solve:

$$\lim_{(x,y)\to(1,2)} \frac{5x^2y}{x^2+y^2} = \frac{5(1)^2(2)}{(1)^2+(2)^2} = \frac{10}{5} = 2$$

Example (2): Calculate the limit $\lim_{(x,y)\to(0,0)} \frac{4x^4-y^4}{2x^2+y^2}$

Solve:

$$\lim_{(x,y)\to(0,0)} \frac{4x^4-y^4}{2x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(2x^2-y^2)(2x^2+y^2)}{2x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} (2x^2-y^2) = 0-0 = 0$$

Example (3): Show that the limit $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist(D.N.E).

Solve: using the path method

A long x=0 $\lim_{(y)\to(0)} \frac{-y^2}{y^2} = \lim_{(y)\to(0)} (-1) = -1$

A long y=0 $\lim_{(x)\to(0)} \frac{x^2}{x^2} = \lim_{(x)\to(0)} (1) = 1$

Since the two paths have different answers, then the limit (D.N.E).

Example (4): Show that the limit $\lim_{(x,y)\to(1,1)} \frac{x-y^4}{x^3-y^4}$ does not exist.

Solve:

A long x=1 $\lim_{(y)\to(1)} \frac{1-y^4}{1-y^4} = \lim_{(y)\to(1)} 1 = 1$

A long y=1 $\lim_{(x)\to(1)} \frac{x-1}{x^3-1} = \lim_{(x)\to(1)} \frac{x-1}{(x-1)(x^2+x+1)} = \lim_{(x)\to(1)} \frac{1}{(x^2+x+1)} = \frac{1}{(1)^2+1+1} = \frac{1}{3}$

Since the two paths have different answers, then the limit (D.N.E).

Example (5): Show that the limit $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

Solve:

A long x=0 $\lim_{(y)\to(0)} \frac{0}{y^2} = 0$

A long y=0 $\lim_{(x)\to(0)} \frac{0}{x^2} = 0$

A long y=x $\lim_{(x,y)\to(0,x)} \frac{x^2}{x^2+x^2} = \lim_{(x,y)\to(0,x)} \frac{x^2}{2x^2} = \lim_{(x,y)\to(0,x)} \frac{1}{2} = \frac{1}{2}$

Since the two paths have different answers, then the limit (D.N.E).

Example (6): Show that the limit $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^3}$ does not exist.

Solve:

A long x=0 $\lim_{(y)\to(0)} \frac{0}{y^4} = 0$

A long y=0 $\lim_{(x)\to(0)} \frac{0}{x^2} = 0$

A long $x=y^2$ $\lim_{(x,y)\to(y^2,0)} \frac{y^4}{y^4+y^4} = \lim_{(x,y)\to(y^2,0)} \frac{y^4}{2y^4} = \lim_{(x,y)\to(y^2,0)} \frac{1}{2} = \frac{1}{2}$

Since the two paths have different answers, then the limit (D.N.E).

Example (7): Calculate the limit $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$.

Solve: using the polar method

$$y = r\sin\theta$$
$$x = r\cos\theta$$
$$r^2 = x^2 + y^2$$

$$\lim_{r\to0} \frac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2} = \lim_{r\to0} \frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2}$$

$$\lim_{r\to0} r(\cos^3\theta - \sin^3\theta) = 0 * (\cos^3\theta - \sin^3\theta) = 0$$ **The limit is exist**

Example (8): Calculate the limit $\lim_{(x,y)\to(0,0)} \frac{yx^2e^x}{x^2+5y^2}$.

Solve: using the polar method

$$y = r\sin\theta$$
$$x = r\cos\theta$$
$$r^2 = x^2 + y^2$$

$$\lim_{r\to0} \frac{(r\sin\theta)(r^2\cos^2\theta)e^{r\cos\theta}}{(r^2\cos^2\theta)+5(r^2\sin^2\theta)} = \lim_{r\to0} \frac{r^3(\sin\theta)(\cos^2\theta)e^{r\cos\theta}}{r^2(\cos^2\theta+5\sin^2\theta)}$$

$$\lim_{r\to0} \frac{r(\sin\theta)(\cos^2\theta)e^{r\cos\theta}}{(\cos^2\theta+5\sin^2\theta)} = \frac{0*(\sin\theta)(\cos^2\theta)e^0}{(\cos^2\theta+5\sin^2\theta)} = \frac{0}{(\cos^2\theta+5\sin^2\theta)} = 0$$ **The limit is exist**

Example (9): $f(x,y) = \begin{cases} \frac{y^2x^2}{2x^2+y^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$ **is $f(x,y)$ continuity or not?**

Solve: lim + =lim - = f(0,0)

$$\lim_{(x,y)\to(0,0)} \frac{y^2x^2}{2x^2+y^2} = \lim_{r\to0} \frac{(r^2\sin^2\theta)(r^2\cos^2\theta)}{2r^2\cos^2\theta + r^2\sin^2\theta^2} = \lim_{r\to0} \frac{r^4(\sin^2\theta)(\cos^2\theta)}{r^2(2\cos^2\theta + \sin^2\theta^2)}$$

$$\lim_{r\to 0}\frac{r^2(sin^2\theta)(cos^2\theta)}{(2cos^2\theta+sin^2\theta^2)}=\frac{0*(sin^2\theta)(cos^2\theta)}{(2cos^2\theta+sin^2\theta^2)}=0$$

$(lim+=0)\neq(lim-=1)$

$f(0,0)=1$

$lim+\neq lim-\neq f(0,0)$ $\quad$ $f(x,y)$ **is not continuity**

Example (10): Discuss the continuity of the function $f(x,y)=\frac{1}{x^2+y^2-4}$

Solve: $\lim_{(x,y)\to(0,0)}=f(0,0)$

$$\lim_{(x,y)\to(0,0)}\frac{1}{x^2+y^2-4}=\frac{1}{0+0-4}=\frac{-1}{4}$$

$$f(0,0)=\frac{1}{0+0-4}=\frac{-1}{4}$$

$\lim_{(x,y)\to(0,0)}=f(0,0)$ $\quad\therefore$ $f(x,y)$ **is continuity**

H.W: Calculate the limit:

1- $\lim_{(x,y)\to(0,0)}\frac{xy^4}{x^2+y^8}$ **2-** $\lim_{(x,y)\to(1,0)}\frac{xy-y}{(x-1)^2+y^2}$ **3-** $\lim_{(x,y)\to(0,0)}\frac{yx^2e^y}{x^4+4y^2}$

4- Discuss the continuity of the function $f(x,y)=\frac{y}{x^2+y^2}$