Example :

Use Kuhn-Tucker conditions to solve:

$$Max \ f(x_1, x_2) = x_1^2 + x_2^2$$
$$s.t. \ g(x_1, x_2) = (x_1 - 2)^2 + x_1 x_2 - 2x_2 = 0$$

<u>Solution</u> :

Our $g(x_1, x_2) = x_1^2 + x_2 - 3 \geq 0$. The Kuhn-Tacker condition are:

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial g}{\partial x_1} = 0 \quad \Rightarrow 2x_1 - \lambda[2x_1 - 4 + x_2] = 0 \quad \text{..........(1)}$$
$$\frac{\partial f}{\partial x_2} - \lambda \frac{\partial g}{\partial x_2} = 0 \quad \Rightarrow 2x_2 - \lambda[x_1 - 2] = 0 \quad \text{..........(2)}$$
$$\lambda g(x_1, x_2) = 0 \quad \Rightarrow \lambda[(x_1 - 2)^2 + x_1 x_2 - 2x_2] = 0 \quad \text{..........(3)}$$
$$g(x_1, x_2) \geq 0 \quad \Rightarrow (x_1 - 2)^2 + x_1 x_2 - 2x_2 = 0 \quad \text{..........(4)}$$
$$\lambda \geq 0 \quad \Rightarrow \lambda \geq 0 \quad \text{..........(5)}$$

معادلات

التأكيد

From (2) we get:

$$\lambda = \frac{2x_2}{x_1 - 2} \quad \text{..........(6)}$$

Putting (6) and (1):

$$2x_1 - \frac{2x_2}{(x_1 - 2)}[2(x_1) - 4 + (x_2)] = 0$$
$$2x_1(x_1 - 2) - 2x_2[2x_1 - 4 + x_2] = 0 \quad \text{..........(7)}$$
$$2x_1^2 - 4x_1 - 4x_1 x_2 + 8x_2 - 2x_2^2 = 0$$

From (3), we get:

$$\frac{2x_2}{(x_1 - 2)}[(x_1 - 2)^2 + x_1 x_2 - 2x_2] = 0 \quad \text{..........(8)}$$

from (8) we get:

$$\frac{2x_2}{(x_1 - 2)} = 0 \quad or \quad (x_1 - 2)^2 + x_1 x_2 - 2x_2 \quad \text{..........(9)}$$

if $\frac{2x_2}{(x_1 - 2)} = 0$ we have,

$$2x_2 = 0 \quad \Rightarrow \quad x_2 = 0 \qquad \text{..........}(10)$$

from (7), we have:

$$2x_1^2 - 4x_1 = 0 \quad \Rightarrow \quad 2x_1^2 = 4x_1 \quad \Rightarrow \quad x_1 = 2 \qquad \text{..........}(11)$$

Then :

$x_1 = 2$, $x_2 = 0$, $f = 4$ satisfies the K.T.C

لاحظ إن هذه النتيجة تحقق المعادلات (4) و (3)

<u>The other possibility:</u>

If :

$$(x_1 - 2)^2 + x_1 x_2 - 2x_2 = 0 \qquad \text{..........}(12)$$

From (12) we have:

$$\frac{(x_1 - 2)^2}{x_2} + x_2(x_1 - 2) = 0 \quad \Rightarrow$$
$$(x_1 - 2) = -x_2 \quad \Rightarrow \quad x_2 = 2 - x_1 \qquad \text{..........}(13)$$

Putting (13) in (7), we obtained:

$$2x_1^2 - 4x_1 - 4x_1(2 - x_1) - 8(2 - x_1) - 2(2 - x_1)^2 = 0 \qquad \text{..........}(14)$$

From (6), we get:

$$2x_1^2 + 4x_1 - 24 = 0 \quad \Rightarrow \quad x_1^2 + x_1 - 6 = 0 \qquad \text{..........}(15)$$

The solution of (15), we obtained:

$$x_1 = \frac{-1 \mp \sqrt{1 + 24}}{2} = \frac{-1 \mp 5}{2} \qquad \text{..........}(16)$$

Putting (16) in (13), we get:

$$\therefore x_1 = 0.5 + 2.5 \quad \Rightarrow \quad x_1 = 2, \quad x_2 = 0 \quad , \quad f = 4. \qquad \text{..........}(16)$$

<u>Excersise</u> :

1) $\begin{aligned} &Max \;\; f(x_1, x_2) = x_1^2 + x_2^2 \\ &s.t. \;\; g(x_1, x_2) = (x_1 - 2)^2 + x_1 x_2 - 2x_2 = 0 \end{aligned}$

2) $\begin{aligned} &Max \;\; f = x_1 x_2 x_3 \\ &s.t. \;\; x_1^2 + x_2^2 + x_3^2 \le 27 \end{aligned}$

3) $\begin{aligned} &Min \;\; f = \frac{1}{x_1} + \frac{1}{x_2} + \frac{x_2^2}{2} + \frac{x_1^2}{2} \\ &s.t. \;\; x_1^2 + x_2^2 \ge 0 \end{aligned}$

Q: *Solve the following nonlinear problem by use the* ***Kuhn – Tucker conditions:***

$$\begin{aligned} &Min \; z = -x_1 - x_2 - x_3 \\ &s.t.: \qquad x_1^2 + x_2 = 3 \\ &\qquad\qquad x_1 + 3x_2 + 2x_3 = 7 \end{aligned}$$

Solve:

$$f_{x_1} - \lambda_1 g_1 - \lambda_2 g_2 = 0 \quad \Rightarrow \quad -1 - 2x_1\lambda_1 - \lambda_2 = 0 \qquad ....(1)$$

$$f_{x_2} - \lambda_1 g_1 - \lambda_2 g_2 = 0 \quad \Rightarrow \quad -1 - \lambda_1 - 3\lambda_2 = 0 \qquad ....(2)$$

$$f_{x_3} - \lambda_1 g_1 - \lambda_2 g_2 = 0 \quad \Rightarrow \quad -1 - 2\lambda_2 = 0 \qquad ....(3)$$

$$\lambda_1 g_1(x_1, x_2, x_3) \quad \Rightarrow \quad \lambda_1(x_1^2 - x_2 - 3) = 0 \qquad ....(4)$$

$$\lambda_2 g_2(x_1, x_2, x_3) \quad \Rightarrow \quad \lambda_2(x_1 + 3x_2 + 2x_3 - 7) = 0 \qquad ....(5)$$

من رقم (3) نحصل على :

$$\lambda_2 = -1/2 \qquad ....(6)$$

من رقم (2) و(8) نحصل على:

$$\lambda_1 = 1/2 \qquad ....(7)$$

نعوض (6) و (7) في (1) نحصل على :

$$x_1 = -1/2 \qquad ....(8)$$

من رقم (4) نحصل على:

أما $x_2 = 11/4$ أو $1/2 = 0$ وهذا غير ممكن ....(9)

من رقم (5) نحصل على :

$x_3 = 3/4$. ....(10)

Q: *Solve the following nonlinear problem by use the* ***Kuhn – Tucker conditions:***

$$Min\ f = x^2 + y^2 + z^2$$
$$s.t.: \quad x + y + z = 1$$

Solve:

$f_x - \lambda g_x = 0 \quad \Rightarrow \quad 2x - \lambda = 0$ ....(1)

$f_y - \lambda g_y = 0 \quad \Rightarrow \quad 2y - \lambda = 0$ ....(2)

$f_z - \lambda g_z = 0 \quad \Rightarrow \quad 2z - \lambda = 0$ ....(3)

$\lambda g(x, y, z) = 0 \quad \Rightarrow \quad \lambda(x + y + z - 1) = 0$ ....(4)

من رقم (1) و (2) و (3) نحصل على :

$\lambda = 2x \quad , \quad \lambda = 2y \quad , \quad \lambda = 2z$ ....(3)

$\therefore \quad 3x = 1 \quad \Rightarrow \quad x = y = z = 1/3$ ....(4)

Q: *Solve the following nonlinear problem by use the* ***Lagranges multipliers:***

$$Min\ f = x\ y$$
$$s.t \quad x^2 + y^2 = 1$$

Solve:

$f_x + \lambda g_x = 0 \quad \Rightarrow \quad y + 2\lambda x = 0$ ....(1)

$f_y + \lambda g_y = 0 \quad \Rightarrow \quad x + 2\lambda y = 0$ ....(2)

من رقم (1) و (2) نحصل على :

$\lambda = -y/2x \quad , \quad \lambda = -x/2y$ ....(3)

$$-y/2x=-x/2y \quad \Rightarrow \quad -2y^2=-2x^2 \qquad ....(4)$$

من رقم (4) نحصل على :

$$x^2 = y^2 \qquad ....(5)$$

إذا :

$$\therefore 2x^2=1 \quad \Rightarrow \quad x^2=1/2 \quad \Rightarrow x=\mp 1/\sqrt{2} \qquad ....(6)$$

ثم نحصل على قيمة y **.**