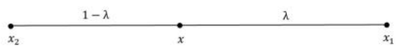
Definition:

A set s is convex if $x_1 \in s$, $x_2 \in s$ $\Rightarrow$ $x = \lambda x_1 + (1-\lambda) x_2 \in s$ for $0 \le \lambda \le 1$.

Geometrically, it means that if the points $A \in s$, $B \in s$, then every point on the line segment $A\,B \in s$.



Note : $x = \lambda x_1 + (1-\lambda) x$ n of x_1 and x_2 .

التحدب (convex) تعني اذا كا خرى مثل B أيضا في s

فالخط الواصل بينهما ينتمي الى s ايضاً.

Theorem :

In a Lpp the set of feasible solution is convex.

الحل الموجود لأي مسألة في البرمجة الخطية هو convex.

Proof :

In general we have:

$$Max\ z = \sum_{j=1}^{n} c_j x_j \qquad ....(1)$$

$$\text{S.t} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i \quad , i = 1,,2,\dots,m \qquad ....(2)$$

$$\forall x_j \ge 0 \, , \, j = 1,2,\dots,m \qquad ....(3)$$

Let x_1 and x_2 two basic feasible solution of this Lpp, then

$$A x_1 = B \, , \quad x_1 \ge 0 \qquad ....(4)$$

$$A x_2 = B \, , \quad x_2 \ge 0 \qquad ....(5)$$

Let $x = \lambda x_1 + (1-\lambda) x_2$ be convex, combination of x_1 and x_2 , then we get:

$$
\begin{aligned}
Ax &= A(\lambda x_1 + (1-\lambda) x_2) \\
&= \lambda A x_1 + (1-\lambda) A x_2 \\
&= \lambda B + (1-\lambda) B \\
&= \lambda B + B - \lambda B \\
&= B
\end{aligned}
$$

$$\therefore \quad Ax = B \qquad \text{........(6)}$$

Hence x is also basic feasible solution, the only remaining thing is to show that $x \geq 0$.

Again $x = \lambda\ x_1 + (1-\lambda)\ x_2$. Since $0 \leq \lambda \leq 1$, λ is (+ive) and also, $x_1 \geq 0$, $x_2 \geq 0$, then:

$$\therefore \quad x \geq 0$$

Hence x is feasible solution and x is convex.

Basic feasible solutions

Consider a system of linear equations $Ax=b$ with m equations and n variables (assume $n \geq m$).

Definition.

A basic solution to a system of linear equations $Ax=b$ is obtained by setting $n-m$ variables equal to 0 and solving for the values of the remaining m variables. This assumes that setting $n-m$ variables equal to 0 yields unique values for the remaining m variables or, equivalently, the columns for the remaining m variables are linearly independent.

The variables that are set to zero are called **non-basic variables**, and the remaining ones are called **basic variables**. If an LP in standard form has m constraints and n variables, then the maximal number of basic solution is :

$$\binom{n}{m} = \frac{n!}{(n-m)!\, m!}$$

Definition.

Any basic solution in which all variables are nonnegative is a **basic feasible solution**. Otherwise, the basic solution is infeasible.

Definition (1):

A feasible solution to Lpp is a vector $(x_1 , x_2 , x_3 ,.........., x_n)$ which satisfies (2) and (3).

Definition (2):

A basic solution of (2) is a solution obtained by setting n-m variables equal to zero and solving for the remaining in variables provided that the determinant of the coefficient of these m variables is non-zero. the m variables are called basic variables.

Definition (3):

A basic feasible solution is a basic solution which also satisfies eq.(3).

i.e., all basic variables are non-negative.

Definition (4):

Anon-degenerate basic feasible solution is a basic feasible solution with exactly m positive x_i .

Example :

$$\begin{aligned} Max \quad z &= x_1 + 2x_2 \\ x_1 - 2x_2 &\le 3 \\ x_1 + x_2 &\le 3 \\ x_i &\ge 0 , \forall i \end{aligned}$$

How to find the initial basic feasible solution (b.f.s) for Lpp:

$$\begin{aligned} &Max \quad z = x_1 + 2x_2 \\ &x_1 - 2x_2 + x_3 = 3 \qquad \ldots\ldots\ldots(1) \\ &x_1 + x_2 + x_4 = 3 \qquad \ldots\ldots\ldots(2) \\ &x_i \geq 0, \forall i \end{aligned}$$

From above we get :

$$n = 4 \quad , \quad M = 2$$

Then,

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = 6$$

The solutions are :

1. $(0,0,3,3)$ B.f.s $z=0$ (√)
2. $(0,-3/2,0,9/2)$ $z=-3$ (×)
3. $(0,3,9,0)$ $z=6$ (√)
4. $(3,0,0,0)$ $z=3$ (√)
5. $(3,0,0,0)$ $z=3$ (√)
6. $(3,0,0,0)$ $z=3$ (√)

$\therefore$ الحل الأمثل $z=3$

***A basic feasible solution is non-degenerate if all of its basic variables are positive.**

***A basic feasible solution is degenerate if some of its basic variables are equal to zero.**

Theorem 3 A point in the feasible region of an LP-problem is an extreme point if and only if it is a basic feasible solution to the LP.