Optimality Conditions.

The condition for the solution x^* of the zero-finding problem is easy to state : x^* is a solution if $f^{/}(x^*) = 0,$.

The conditions for the solution of the minimum-finding problem are not so simple.

Conditions in 1-Dimension. From elementary calculus we know that for a 1-dimensional function the following conditions must hold at an optimum $x^* \in R^n$:

$$f^{/}(x^*) = 0,$$

$f^{//}(x^*) < 0$, at a maximum;

$f^{//}(x^*) < 0$, at a minimum;

$f^{//}(x^*) = 0$, at a point of inflection.

Conditions in n-Dimensions. The equivalent conditions in **n** dimensions are :

$$f^{/}(x^*) = 0,$$

$G = \nabla^2 f(x)$, is negative definite at a maximum;

$G = \nabla^2 f(x)$, is positive definite at a minimum;

$G = \nabla^2 f(x)$, is indefinite at a saddle point.

where $G = \nabla^2 f(x)$ is symmetric.

Convexity of Optimization Problems

In what follows we define convex set and convex function.

Definition :

A set of points C is called convex set if for all $0 \le \lambda \le 1$, $\lambda x + (1-\lambda) y$ is contained in C, when every x and y are contained in C.

Definition :

A function $f(x)$ is called **convex** if for every x , y and every $0 \le \lambda \le 1$, we have :

$$f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y) .$$

Also, it is called **strictly convex** if for every x , y and every $0 \le \lambda \le 1$, we have :

$$f(\lambda x + (1-\lambda) y) < \lambda f(x) + (1-\lambda) f(y) .$$

Definition :

A function $f(x)$ is called **concave** if for every x , y and every $0 \le \lambda \le 1$, we have :

$$f(\lambda x + (1-\lambda) y) \ge \lambda f(x) + (1-\lambda) f(y) .$$

Also, it is called **strictly concave** if for every x , y and every $0 \le \lambda \le 1$, we have :

$$f(\lambda x + (1-\lambda) y) > \lambda f(x) + (1-\lambda) f(y) .$$

Basic properties of convex functions

In this section we have collected some useful facts about convex functions.

- If $f(x)$ is convex function, its sublevel set $f(x) \le \alpha$ is convex.
- Positive multiple of convex function is convex :

  $f(x)$ is convex, $\alpha \ge 0 \quad \Rightarrow \quad \alpha f(x)$ convex.
- Sum of convex functions is convex :

  $f_1(x)$, $f_2(x)$ are convex, $\quad \Rightarrow \quad f_1(x) + f_2(x)$ convex.
- Point wise maximum of convex functions is convex :

$f_1(x)$, $f_2(x)$ are convex, $\quad \Rightarrow \quad Max\{f_1(x), f_2(x)\}$ convex.

A **function** is increasing if $f'(x) > 0$, decreasing if $f'(x) < 0$, and neither if $f'(x) = 0$.

- Composite function :

$$f(x) = h(g(x))$$

is convex if :

1. g convex; h convex nondecreasing
2. g concave; h convex nonincreasing

The following definition can be used when function $f(x)$ is differentiable.

Using second derivative :

1. A function is strictly c**onvex** if $f''(x) > 0$ and strictly **concave** if $f''(x) < 0$.
2. A function is **convex** if $f''(x) \geq 0$ and **concave** if $f''(x) \leq 0$.

Note :

If $f(x)$ is convex, then any local minimum is also a global minimum.

If $f(x)$ is concave, then any local maximum is also a global maximum.

Example :

Show that the function $f(x) = -\ In\ x$ is a strictly convex function.

Solution :

The first and second-order partial derivatives of $f(x)$ are given by :

$$f'(x) = -\ 1/x \text{ and } f''(x) = 1/x^2 > 0$$

$f(x)$ is strictly convex function.

Example :

Show that the function $f(x) = x_1^2 + x_2 + x_3^2 - 10$ is a strictly convex function.

Solution :

The first and second-order partial derivatives of $f(x)$ are given by :

$$g = f^{/}(x) = \begin{bmatrix} 2x_1 \\ 1 \\ 2x_3 \end{bmatrix} \text{ and } G = f^{//}(x) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \geq 0$$

Hessian is positive definite and so the function $f(x)$ is convex.