Total differential, chain rule & Extrema of Functions of Two Variables

التفاضل الكلي ، قاعدة السلسلة و الحدود القصوى للدوال ذات المتغيرين

Definitions and Theorems: **تعاريف ونظريات**

- Let $Z = \boldsymbol{f(x,y)}$ then the total differential of z is the expression

لتكن الدالة $Z = \boldsymbol{f(x,y)}$ فإن التفاضل الكلي لـ z يعبر عنه

$$dZ = \frac{\partial Z}{\partial x}\mathrm{dx} + \frac{\partial Z}{\partial y}\mathrm{dy} = f_x(\boldsymbol{x},\boldsymbol{y})dx + f_y(\boldsymbol{x},\boldsymbol{y})dy$$

- Let w be a function of x and y, and assume that x and y are both functions of t. Then, w is a function of t, and the **chain rule** says that

لتكن w دالة لـ x و y، وافترض أن x و y كلاهما دالة لـ t. فأن w هي دالة لـ t، وتنص قاعدة السلسلة على أنها:

$$\boldsymbol{\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}}$$

- Let $\boldsymbol{f(x,y)}$ be a function defined on the region R containing $(\boldsymbol{x_0},\boldsymbol{y_0})$, The point $(\boldsymbol{x_0},\boldsymbol{y_0})$ is **critical point** if either:

لتكن $\boldsymbol{f(x,y)}$ دالة معرفة على المنطقة R التي تحتوي على $(\boldsymbol{x_0},\boldsymbol{y_0})$. النقطة $(\boldsymbol{x_0},\boldsymbol{y_0})$ هي نقطة حرجة إذا كان أي من النقطتين تحقق:

1) $f_x(\boldsymbol{x_0},\boldsymbol{y_0}) = 0 \ \ and \ f_y(\boldsymbol{x_0},\boldsymbol{y_0}) = 0$
2) $f_x(\boldsymbol{x_0},\boldsymbol{y_0}) \ or \ f_y(\boldsymbol{x_0},\boldsymbol{y_0}) \ do \ not \ exist$

- Relative extrema occur at critical points. In other words, the critical points are the candidates for relative maximum and relative minimum. This is done through the following second partials test:

تحدث القيم القصوى النسبية عند النقاط الحرجة. بعبارة أخرى، النقاط الحرجة هي المرشحة للقيم العظمى (الحد الاعلى) النسبية والقيم الصغرى (الحد الادنى) النسبية. ويتم ذلك من خلال الاختبار الجزئي الثاني التالي:

$$D = f_{xx}(\boldsymbol{x_0},\boldsymbol{y_0})f_{yy}(\boldsymbol{x_0},\boldsymbol{y_0}) - (f_{xy}(\boldsymbol{x_0},\boldsymbol{y_0}))^2$$

Then, we have the following:

1) $D > 0 \ , f_{xx}(\boldsymbol{x_0},\boldsymbol{y_0}) > 0 \quad \rightarrow relative \ minimum$
2) $D > 0 \ , f_{xx}(\boldsymbol{x_0},\boldsymbol{y_0}) < 0 \quad \rightarrow relative \ maximum$
3) $D < 0 \qquad \rightarrow saddle \ point$
4) $D = 0 \qquad \rightarrow Test \ is \ inconclusive$ الاختبار غير حاسم

The second partials test can fail in two ways: قد يفشل اختبار الجزئيات الثاني بطريقتين

1) The partial derivatives might not exist. قد لا توجد المشتقات الجزئية
2) The discriminant D=0.

Example (1): Find the total differential of the function $\boldsymbol{z = 2xsin(y) - 3x^2y^2}$

Solve: $\boldsymbol{dZ = \frac{\partial Z}{\partial x}dx + \frac{\partial Z}{\partial y}dy}$

$\boldsymbol{\frac{\partial Z}{\partial x} = 2sin(y) - 6xy^2}$ $\qquad$ $\boldsymbol{\frac{\partial Z}{\partial y} = 2xsin(y) - 6x^2y}$

$\boldsymbol{dZ = (2sin(y) - 6xy^2)dx + (2xcos(y) - 6x^2y)dy}$

Example (2): Find the total differential of the function $\boldsymbol{w = x^2 + y^3 + 4z^4}$

Solve: $\boldsymbol{dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz}$

$$\boldsymbol{\frac{\partial w}{\partial x} = 2x \qquad \frac{\partial w}{\partial y} = 3y^2 \qquad \frac{\partial w}{\partial z} = 16z^3}$$

$$\boldsymbol{dw = 2xdx + 3y^2dy + 16z^3dz}$$

Example (3): Find the total differential of the function $\boldsymbol{z = 2x^2y^3}$

Solve: $\boldsymbol{dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy}$

$$\boldsymbol{\frac{\partial z}{\partial x} = 4xy^3 \qquad \frac{\partial w}{\partial y} = 6x^2y^2}$$

$$\boldsymbol{dz = 4xy^3dx + 6x^2y^2dy}$$

Example (4): Use the chain rule to find $\boldsymbol{\frac{dw}{dt}}$ if $\boldsymbol{w = x^2y - y^2 , x = \sin(t)\ and\ y = e^t}$

Solve: $\boldsymbol{\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}}$

$$\frac{\partial w}{\partial x} = 2yx \quad , \quad \frac{\partial w}{\partial y} = x^2 - 2y$$

$$\frac{dx}{dt} = cos(t) \quad , \quad \frac{dy}{dt} = e^t$$

$$\frac{dw}{dt} = 2yx(cost) + (x^2 - 2y)(e^t)$$

$$\frac{dw}{dt} = 2e^t sint(cost) + ((sint)^2 - 2e^t)(e^t)$$

$$\frac{dw}{dt} = 2e^t sint(cost) + e^t(sint)^2 - 2e^{2t}$$

Example (5): Use the chain rule to find $\boldsymbol{\frac{dw}{dt}}$ if $\boldsymbol{w = xy , x = e^t\ and\ y = e^{-2t}}$

Solve: $\boldsymbol{\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}}$

$$\frac{\partial w}{\partial x} = y \quad , \quad \frac{\partial w}{\partial y} = x$$

$$\frac{dx}{dt} = e^t \quad , \quad \frac{dy}{dt} = -2e^{-2t}$$

$$\frac{dw}{dt} = y(e^t) + x(-2\boldsymbol{e^{-2t}}) = \boldsymbol{e^{-2t}}e^t - 2e^t\boldsymbol{e^{-2t}}$$

$$\frac{dw}{dt} = e^{-t} - 2e^{-t} = -e^{-t}$$

Example (6): Use the chain rule to find $\frac{dw}{dt}$ if $\boldsymbol{w = e^{xy}}$, $\boldsymbol{x = t^2}$ **and** $\boldsymbol{y = t}$

Solve: $\boldsymbol{\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}}$

$\frac{\partial w}{\partial x} = ye^{xy}$, $\frac{\partial w}{\partial y} = xe^{xy}$

$\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 1$

$$\frac{dw}{dt} = ye^{xy}(2t) + xe^{xy}(1)$$

$$\frac{dw}{dt} = te^{tt^2}(2t) + t^2e^{tt^2} = 2t^2e^{t^3} + t^2e^{t^3} = 3t^2e^{t^3}$$

Example (7): Find the critical point and determine the relative extrema of the function $\boldsymbol{f(x,y) = 2x^2 + y^2 + 8x - 6y + 20}$

Solve:

$f_x(x,y) = 4x + \mathbf{8}$

$0 = 4x + \mathbf{8} \rightarrow \mathbf{4x = -8} \rightarrow \boldsymbol{x = -2}$

$f_y(x,y) = 2y - \mathbf{6}$

$0 = 2y - \mathbf{6} \rightarrow \mathbf{2y = 6} \rightarrow \boldsymbol{y = 3}$

$(\boldsymbol{x_0}, \boldsymbol{y_0}) = (-2,3)$

$\because f_x(\boldsymbol{x_0}, \boldsymbol{y_0}) = 4(-2) + 8 = -8 + 8 = 0$ $and\ f_y(\boldsymbol{x_0}, \boldsymbol{y_0}) = 2(3) - 6 = 6 - 6 = 0$

$\therefore$ (-2,3) is the critical point.

$D = f_{xx}(\boldsymbol{x_0}, \boldsymbol{y_0})f_{yy}(\boldsymbol{x_0}, \boldsymbol{y_0}) - (f_{xy}(\boldsymbol{x_0}, \boldsymbol{y_0}))^2$

$f_{xx}(x,y) = 4$

$f_{yy}(x,y) = 2$

$f_{xy}(x,y) = 0$

$D = (4)(2) - (0)^2 = 8$

$\because D > 0$, $f_{xx}(\boldsymbol{x_0}, \boldsymbol{y_0}) > 0$ $\rightarrow relative\ minimum$

Example (8): Find the critical point and determine the relative extrema of the function $\boldsymbol{f(x,y) = -5x^2 + 4xy - y^2 + 16x + 10}$

Solve:

$f_x(x,y) = -10x + \mathbf{4y + 16}$

$0 = -10x + \mathbf{4y + 16}$ ... $(\mathbf{1})$

$f_y(x,y) = \mathbf{4x - 2y}$

$0 = \mathbf{4x - 2y}$ ... $(\mathbf{2})$

Milt Eq.(2) by (2)

$0 = -10x + \mathbf{4y} + \mathbf{16} \qquad \ldots(\mathbf{1})$

$\underline{0 = \quad \mathbf{8x} - \mathbf{4y} \qquad \ldots(\mathbf{2})}$ بالجمع

$0 = -2x + \mathbf{16} \quad \rightarrow \quad 2x = \mathbf{16} \quad \rightarrow \quad x = \mathbf{8}$

Sub. X=8 in Eq.1

$0 = -10(8) + \mathbf{4y} + \mathbf{16} \;\; = -\mathbf{80} + \mathbf{4y} + \mathbf{16}$

$0 = -64 + 4y \;\rightarrow\; 4y = \mathbf{64} \quad \rightarrow \; y = \mathbf{16}$

$(\boldsymbol{x_0}, \boldsymbol{y_0}) = (8,16)$

$\because f_x(\boldsymbol{x_0}, \boldsymbol{y_0}) = -10(8) + \mathbf{4}(\mathbf{16}) + \mathbf{16} = -80 + 64 + 16 = 0$

$and \; f_y(\boldsymbol{x_0}, \boldsymbol{y_0}) = \mathbf{4}(\mathbf{8}) - \mathbf{2}(\mathbf{16}) = 32 - 32 = 0$

∴ (8,16) is the critical point.

$D = f_{xx}(\boldsymbol{x_0}, \boldsymbol{y_0}) f_{yy}(\boldsymbol{x_0}, \boldsymbol{y_0}) - \left(f_{xy}(\boldsymbol{x_0}, \boldsymbol{y_0})\right)^2$

$f_{xx}(x, y) = -10$

$f_{yy}(x, y) = -2$

$f_{xy}(x, y) = 4$

$D = (-10)(-2) - (4)^2 = 4$

$\because D > 0 \;, f_{xx}(\boldsymbol{x_0}, \boldsymbol{y_0}) < 0 \quad \rightarrow relative\ maximum$

Example (9): Find the critical point and determine the relative extrema of the function

$\boldsymbol{f(x, y) = y^2 - x^2}$

Solve:

$f_x(x, y) = -\mathbf{2x}$

$0 = -\mathbf{2x} \quad \rightarrow \; \boldsymbol{x = 0}$

$f_y(x, y) = \mathbf{2y}$

$0 = \mathbf{2y} \quad \rightarrow \; \boldsymbol{y = 0}$

$(\boldsymbol{x_0}, \boldsymbol{y_0}) = (0,0)$

$\because f_x(\boldsymbol{x_0}, \boldsymbol{y_0}) = -2(0) = 0 \;\; and \; f_y(\boldsymbol{x_0}, \boldsymbol{y_0}) = 2(0) = 0$

∴ (0,0) is the critical point.

$D = f_{xx}(\boldsymbol{x_0}, \boldsymbol{y_0}) f_{yy}(\boldsymbol{x_0}, \boldsymbol{y_0}) - \left(f_{xy}(\boldsymbol{x_0}, \boldsymbol{y_0})\right)^2$

$f_{xx}(x, y) = -2$

$f_{yy}(x, y) = 2$

$f_{xy}(x, y) = 0$

$D = (-2)(2) - (0)^2 = -4$

$\because D < 0 \qquad \rightarrow saddle\ point$

Example (10): Find the critical point and determine the relative extrema of the function

$$\boldsymbol{f(x,y) = xy}$$

Solve:

$f_x(x,y) = \boldsymbol{y}$

$0 = \boldsymbol{y} \quad \rightarrow \boldsymbol{y = 0}$

$f_y(x,y) = \boldsymbol{x}$

$0 = \boldsymbol{x} \quad \rightarrow \boldsymbol{x = 0}$

$(\boldsymbol{x_0}, \boldsymbol{y_0}) = (0,0)$

$\because f_x(\boldsymbol{x_0}, \boldsymbol{y_0}) = 0 \ \ and \ f_y(\boldsymbol{x_0}, \boldsymbol{y_0}) = 0$

$\therefore$ (0,0) is the critical point.

$D = f_{xx}(\boldsymbol{x_0}, \boldsymbol{y_0}) f_{yy}(\boldsymbol{x_0}, \boldsymbol{y_0}) - (f_{xy}(\boldsymbol{x_0}, \boldsymbol{y_0}))^2$

$f_{xx}(x,y) = 0$

$f_{yy}(x,y) = 0$

$f_{xy}(x,y) = 0$

$D = (0)(0) - (0)^2 = 0$

$\because D = 0 \quad \rightarrow Test\ is\ inconclusive$

H.W:

1- Find the critical point and determine the relative extrema of the function

$$\boldsymbol{f(x,y) = 3x^2 + 2y^2 - 6x - 4y + 16}$$

2- Find the critical point and determine the relative extrema of the function

$$\boldsymbol{f(x,y) = 2x + 4y - x^2 - y^2}$$