Lagrange multipliers & Applications of Lagrange Multipliers
مضاعفات لاكرانج و تطبيقات مضاعفات لاكرانج

- **Lagrange's Theorem:** Let f & g have continuous first partial derivatives such that f has an extremum at (x_0, y_0) on the smooth constraint curve $g(x,y) = k$. if $\nabla g(x_0, y_0) \neq 0$ then there is a real number (λ) such that:

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

The number (λ) is called a Lagrange multipliers

نظرية لاكرانج: ليكن لـ f و g لهما مشتقات جزئية أولى متصلة بحيث يكون لـ f حد أقصى عند (x_0, y_0) على منحنى المقيد g(x,y)=k . فإذا كان $\nabla g(x_0, y_0) \neq 0$ فإن هناك عددًا حقيقيًا (λ) يسمى مضاعفات لاكرانج.

ملاحظة: تستخدم مضاعفات لاكرانج لإيجاد القيم العظمى والصغرى من خلال استخدام المشتقات الجزئية فعندما يكون لدينا ثلاث متغيرات (x,y,z) سيتم الحصول على اربع معادلات بعدد المتغيرات المطلوب تقديرها أي (x, y, z, λ) وتكون كالاتي:

$$f_x = \lambda g_x \qquad \dots.(1)$$
$$f_y = \lambda g_y \qquad \dots.(2)$$
$$f_z = \lambda g_z \qquad \dots.(3)$$
$$g(x,y,z) = 0 \qquad \dots.(4)$$

ويتم حل هذه المعادلات أعلاه انياً لتقدير القيم الأربعة (x, y, z, λ) من خلال تقسيم معادلة (1) على (2) وبعدها نقسم المعادلة (1) على (3) وتعويضهم بالمعادلة (4) و هكذا يتم إيجاد القيم (x,y,z) اما λ يتم ايجادها من خلال تعويض قيم (x,y,z).

Example (1): Use Lagrange multipliers to find the minimum value of $f(x,y) = x^2 + y^2$ subject to the constraint $x + 2y - 5 = 0$. Assume that x & y are positive.

Solve:

$\nabla f(x,y) = \lambda \nabla g(x,y)$

$\nabla f(x,y) = \langle f_x , f_y \rangle = \langle 2x , 2y \rangle$

$\nabla g(x,y) = \langle g_x , g_y \rangle = \langle 1 , 2 \rangle$

$\langle 2x , 2y \rangle = \lambda \langle 1 , 2 \rangle$

$2x = \lambda \quad \dots(1)$

$2y = 2\lambda \quad \dots(2)$

$x + 2y - 5 = 0 \quad \dots(3)$

Divit. eq.(1) on eq.(2)

$\frac{2x}{2y} = \frac{\lambda}{2\lambda} \quad \Rightarrow \quad \frac{x}{y} = \frac{1}{2}$

$y = 2x$

Sub y in eq.(3)

$x + 2(2x) - 5 = 0 \Rightarrow x + 4x - 5 = 0 \quad \Rightarrow \quad 5x = 5 \quad \Rightarrow x = 1$

Sub x in y

$y = 2(1) \Rightarrow y = 2$

$2(1) = \lambda \Rightarrow \lambda = 2$

$(x_0, y_0) = (1,2)$

The minimum value is $f(1,2) = 1^2 + 2^2 = 1 + 4 = 5$

Example (2): Use Lagrange multipliers to find the maximum value of $f(x, y) = 4xy$ subject to the constraint $g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} - 1$. Assume that x & y are positive.

Solve:

$\nabla f(x, y) = \langle f_x , f_y \rangle = \langle 4y , 4x \rangle$

$\nabla g(x, y) = \langle g_x , g_y \rangle = \langle \frac{2x}{9} , \frac{2y}{16} \rangle$

$\langle 4y , 4x \rangle = \lambda \langle \frac{2x}{9} , \frac{2y}{16} \rangle$

$4y = \lambda \frac{2x}{9}$ ... (1)

$4x = \lambda \frac{2y}{16}$ ... (2)

$\frac{x^2}{9} + \frac{y^2}{16} - 1 = 0$ ... (3)

Divit eq.(1) on eq.(2)

$\frac{4y}{4x} = \frac{\lambda \frac{2x}{9}}{\lambda \frac{2y}{16}} \Longrightarrow \frac{y}{x} = \frac{16x}{9y} \Longrightarrow 16x^2 = 9y^2 \Longrightarrow x^2 = \frac{9}{16} y^2$

Sub x^2 in eq.(3)

$\frac{\frac{9}{16} y^2}{9} + \frac{y^2}{16} - 1 = 0 \Rightarrow \frac{y^2}{16} + \frac{y^2}{16} - 1 = 0 \Rightarrow \frac{2y^2}{16} = 1$

$2y^2 = 16 \Rightarrow y^2 = 8 \Rightarrow y = 2\sqrt{2}$

Sub y in x^2

$x^2 = \frac{9}{16} (2\sqrt{2})^2 = \frac{9(8)}{16} = \frac{9}{2} \Rightarrow x = \frac{3}{\sqrt{2}}$

Sub x & y in eq.(1)

$4(2\sqrt{2}) = \lambda \frac{2\left(\frac{3}{\sqrt{2}}\right)}{9} \Longrightarrow 8\sqrt{2} = \lambda \frac{\sqrt{2}}{3} \Longrightarrow \lambda = 24$

$(x_0, y_0) = (\frac{3}{\sqrt{2}}, 2\sqrt{2})$

The maximum value is $f\left(\frac{3}{\sqrt{2}}, 2\sqrt{2}\right) = 4\left(\frac{3}{\sqrt{2}}\right)(2\sqrt{2}) = 24$

Example (3): Use Lagrange multipliers to find the minimum value of $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint $g(x,y,z) = x + y + z - 9$. Assume that x , y & z are positive.

Solve:

$\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle = \langle 2x, 2y, 2z \rangle$

$\nabla g(x,y,z) = \langle g_x, g_y, g_z \rangle = \langle 1, 1, 1 \rangle$

$\langle 2x, 2y, 2z \rangle = \lambda \langle 1, 1, 1 \rangle$

$2x = \lambda \quad \ldots (1)$

$2y = \lambda \quad \ldots (2)$

$2z = \lambda \quad \ldots (3)$

$x + y + z - 9 = 0 \quad \ldots (4)$

Divit eq.(1) on eq.(2)

$\frac{2x}{2y} = \frac{\lambda}{\lambda} \quad \Rightarrow \quad y = x$

Divit eq.(1) on eq.(3)

$\frac{2x}{2z} = \frac{\lambda}{\lambda} \quad \Rightarrow \quad z = x$

Sub y & z in eq.(4)

$x + x + x - 9 = 0 \quad \Rightarrow 3x - 9 = 0 \Rightarrow \quad 3x = 9 \quad \Rightarrow x = 3$

Sub x in y & z

$y = 3$, $z = 3$

Sub x in eq.(1)

$2(3) = \lambda \quad \Rightarrow \lambda = 6$

$(x_0, y_0, z_0) = (3,3,3)$

The minimum value is $f(3,3,3) = 3^2 + 3^2 + 3^2 = 9 + 9 + 9 = 27$

Example (4): Use Lagrange multipliers to find the extreme value of $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$

Solve:

$g(x,y) = x^2 + y^2 - 1$

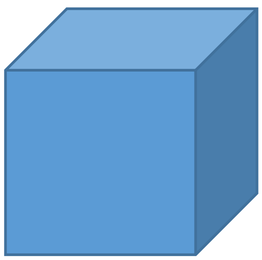
$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle 2x, 4y \rangle$

$\nabla g(x,y) = \langle g_x, g_y \rangle = \langle 2x, 2y \rangle$

$\langle 2x , 4y \rangle = \lambda\langle 2x, 2y\rangle$
$2x = 2\lambda x \qquad ...(1)$
$4y = 2\lambda\, y \qquad ...(2)$
$x^2 + y^2 - 1 = 0 \qquad ...(3)$
Divit eq.(1) on eq.(2)
$\frac{2x}{4y} = \frac{2\lambda x}{2\lambda y} \Rightarrow \frac{x}{2y} = \frac{x}{y} \Rightarrow yx = 2xy \Rightarrow 0 = 2xy - xy \Rightarrow xy = 0$
$\Rightarrow y = 0 \; or \; x = 0 \quad \& \, \lambda = 0$
Sub $y = 0$ in eq.(3)
$x^2 + 0^2 - 1 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \mp 1$
Sub $x = 0$ in eq.(3)
$0^2 + y^2 - 1 = 0 \Rightarrow y^2 = 1 \Rightarrow y = \mp 1$
$(x_0, y_0) = (-1,0) , (1,0) , (0,1) \; \& \; (0,-1)$
$f(1,0) = 1^2 + 2(0)^2 = 1$
$f(-1,0) = (-1)^2 + 2(0)^2 = 1$
$f(0,1) = (0)^2 + 2(1)^2 = 2$
$f(0,-1) = (0)^2 + 2(-1)^2 = 2$
The minimum value is $f(-1,0) = f(1,0) = 1$ .The maximum value is $f(0,1) = f(0,-1) = 2$

Example (5): find the dimensions of the box with volume 1000 cm^3 that minimum surface area.
أوجد أبعاد الصندوق الذي حجمه 1000 سم^3 والذي له أقل مساحة سطحية.

Solve:



$V = xyz = 1000$
$g(x,y,z) = xyz - 1000$
$f(x,y,z) = 2xy + 2yz + 2xz$
$\nabla f(x,y,z) = \langle f_x , f_y, f_z\rangle = \langle 2y + 2z , 2x + 2z , 2y + 2x\rangle$
$\nabla g(x,y,z) = \langle g_x , g_y, g_z\rangle = \langle yz , xz, xy\rangle$
$\langle 2y + 2z , 2x + 2z , 2y + 2x\rangle = \lambda\langle yz , xz, xy\rangle$
$2y + 2z = \lambda\, yz \qquad ...(1)$
$2x + 2z = \lambda xz \qquad ...(2)$
$2y + 2x = \lambda xy \qquad ...(3)$
$xyz - 1000 = 0 \qquad ...(4)$
Divit eq.(1) on eq.(2)
$\frac{2y + 2z}{2x + 2z} = \frac{\lambda\, yz}{\lambda xz} \Rightarrow \frac{y + z}{x + z} = \frac{y}{x} \Rightarrow yx + xz = xy + yz \Rightarrow y = x$
Divit eq.(1) on eq.(3)

$\frac{2y+2z}{2y+2x} = \frac{\lambda\, yz}{\lambda xy} \quad \Rightarrow \frac{y+z}{y+x} = \frac{z}{x} \quad \Rightarrow \quad yx + zx = yz + xz \quad \Rightarrow z = x$

Sub y & z in eq.(4)

$x(x)(x) - 1000 = 0 \quad \Rightarrow \quad x^3 = 1000 \quad \Rightarrow x = 10 \qquad \therefore \ y = 10 \ \& \ z = 10$

$2(10) + 2(10) = \lambda\,(10)(10) \ \Rightarrow 40 = \ 100\lambda \quad \Rightarrow \ \lambda = \frac{4}{10}$

The dimensions of the box (10, 10 , 10)

Sub y & z in eq.(1)

The minimum surface area is

$f(10,10,10) = 2(10)(10) + 2(10)(10) + 2(10)(10) = 200 + 200 + 200 = 600$

Example (6): find 3 positive numbers whose sum is 100 and whose product is maximum.

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Solve:

$x + y + z = 100$

$g(x,y,z) = x + y + z - 100$

$f(x,y,z) = xyz$

$\nabla f(x,y,z) = \langle f_x\,, f_y, f_z \rangle = \langle yz\,, xz\,, xy \rangle$

$\nabla g(x,y,z) = \langle g_x\,, g_y, g_z \rangle = \langle 1\,,1,1 \rangle$

$\langle yz\,, xz\,, xy \rangle = \lambda \langle 1\,,1,1 \rangle$

$yz \ = \lambda \qquad \dots(1)$

$xz = \lambda \qquad \dots(2)$

$xy = \lambda \qquad \dots(3)$

$x + y + z - 100 = 0 \qquad \dots(4)$

Divit eq.(1) on eq.(2) $\ \frac{yz}{xz} = \frac{\lambda}{\lambda} \quad \Rightarrow \frac{yz}{xz} = 1 \ \Rightarrow \ yz = xz \quad \Rightarrow y = x$

Divit eq.(1) on eq.(3) $\ \frac{yz}{xy} = \frac{\lambda}{\lambda} \quad \Rightarrow \frac{yz}{xy} = 1 \ \Rightarrow \ yz = xy \quad \Rightarrow z = x$

Sub y & z in eq.(4)

$x + x + x - 100 = 0 \quad \Rightarrow \ 3x = 100 \ \Rightarrow x = \frac{100}{3} \qquad \therefore \ y = \frac{100}{3} \ \& \ z = \frac{100}{3}$

Sub y & z in eq.(1) $\qquad \left(\frac{100}{3}\right)\left(\frac{100}{3}\right) = \lambda \quad \Rightarrow \quad \lambda = \frac{10000}{9}$

The 3 positive numbers $\left(\frac{100}{3}, \frac{100}{3}, \frac{100}{3}\right)$

The maximum product is

$f\left(\frac{100}{3}, \frac{100}{3}, \frac{100}{3}\right) = \frac{100}{3}\left(\frac{100}{3}\right)\left(\frac{100}{3}\right) = \frac{1000000}{27} = 37037.04$