Differential Equations

1. Definition of Differential Equations

Differential Equation: is any equation which contains one derivative or more. The derivative may be either ordinary derivative or partial derivative.

المعادلة التفاضلية: هي المعادلة التي تحتوي على مشتقة واحدة على الاقل او أكثر، وتكون المشتقة اما مشتقة اعتيادية او مشتقة جزئية.

$$\acute{\acute{y}} + 4\acute{y} - 3y = 2x \quad or \quad \frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 3y = 2x$$

Solution of the Differential Equation is finding an equation without any derivatives, and when substituted in the differential equation achieved.

حل المعادلة التفاضلية هو إيجاد معادلة خالية من المشتقات وإذا عوضت في المعادلة التفاضلية تحققها.

1.1 Classification of Differential equation:

Differential equation divided into:

1) Ordinary Differential Equation: independent variable = 1

Ordinary Differential Equation (ODE): is an equation containing one independent variable.

المعادلة التفاضلية الاعتيادية : وهي المعادلة التي تحتوي على متغير مستقل واحد.

For Example:

$$y = x^2$$

$$\frac{dy}{dx} = 2x \quad or \quad \acute{y} = 2x \quad \rightarrow$$

$$y = f(x)$$

$where$:

y: is dependent variable متغير معتمد

x: is independent variable متغير مستقل

2) Partial Differential Equation: independent variable > 1

Partial Differential Equation (PDE): is an equation containing more than one independent variable.

المعادلة التفاضلية الجزئية :وهي المعادلة التي تحتوي على أكثر من متغير مستقل.

For Example:

$$z = x^2 + y^2$$

$$\frac{dz}{dx} = 2x \quad or \quad \acute{z} = 2x \quad \rightarrow \quad z = f(x, y)$$

$$\frac{dz}{dy} = 2y \quad or \quad \acute{z} = 2y \quad \rightarrow \quad z = f(x, y)$$

$where$:
z: is dependent variable
x & y: is independent variable

1.2 Order of differential equation: **رتبة المعادلات التفاضلية**

The number of highest derivative in a differential equation. A differential equation of order 1 is called First order, order 2 is called Second order ……. etc.

هي اعلى مشتقة موجودة في المعادلة التفاضلية.

For Example:

$\acute{y} + y = x \qquad First\ order$

$\acute{\acute{y}} + 2\acute{y} - y = \sin x \qquad Second\ order$

1.3 Degree of the differential equation: **درجة المعادلة التفاضلية**

The highest power which is raised to the highest-order derivative existed in differential equation.

هو اعلى اس موجود على اعلى مشتقة في المعادلة التفاضلية.

For Example:

1) $\acute{\acute{y}} - 6\acute{y} + 9y = 0 \qquad the\ degree\ is\ 1$

2) $\dddot{y} + 3\ddot{y} + \ln x = 5 \qquad the\ degree\ is\ 1$

3) $\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} + \tan x = 0 \qquad the\ degree\ is\ 3$

4) $\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx} = 0 \qquad the\ degree\ is\ 1$

Note: if highest-order derivative found inside the root or any form of fraction, the degree of this differential equation cannot be determined (undefined degree).

اذا كانت اعلى مشتقة موجودة داخل أي دالة مثل الجذر $\sqrt{\acute{y}}$ او أي دالة مثلثية $\sin(\acute{y})$ او حاصل ضرب $\acute{y}y$ فلا نستطيع تحديد درجة المعادلة التفاضلية.

For example:

$$\sqrt{\frac{d^2y}{dx^2}} + \frac{dy}{dx} = x \quad \text{(undefined degree)}$$

$$\frac{d^2y}{dx^2} + \cos\left(\frac{d^3y}{dx^3}\right) = 5x \quad \text{(undefined degree)}$$

1.4 Types of Differential Equation

There are two types of differential equation according to degree it:

a) Linear Differential Equation: is a differential equation without nonlinear term and all of its terms are of first degree.

b) Non-Linear Differential Equation: is a differential equation that contain nonlinear terms such as: $\sin y$, e^y , $\sqrt{y}$, y^2, $y\acute{y}$ *or* $\ln y$.

كي تصبح المعادلة خطية (linear) يجب ان يكون كل حد من حدود المعادلة الذي يحتوي على المتغير المعتمد(y) ومشتقاته من الدرجة الأولى، أي يجب ان لا يظهر أحد حدود المعادلة على شكل مثلا $\sin y$, e^y , $\sqrt{y}$, y^2, $y\acute{y}$ *or* $\ln y$.

For example:

1) $\acute{\acute{y}} + y\acute{y} = x \quad (Non-Linear\ because\ y\acute{y})$

2) $\left(\frac{d^2y}{dx^2}\right)^3 + x\frac{dy}{dx} = \sin x \quad \left(Non-Linear\ because\ \left(\frac{d^2y}{dx^2}\right)^3\right)$

3) $\acute{\acute{y}} + 4x\acute{y} + 2y = \cos x \quad (Linear\ Equation)$ لا تعتمد على x

4) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + \ln y = 0 \quad (Non-Linear\ because\ \ln y)$

5) $\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 + \tan y = 4 \quad \left(Non-Linear\ because\ \left(\frac{dy}{dx}\right)^2\ \&\ \tan y\right)$

1.5 Solution of the differential equation: **حل المعادلات التفاضلية**

There is multi-solution for the differential equation:

1. General Solution: it is a solution that contain one or more essential constant.

الحل العام :هو ذلك الحل الذي يحتوي على ثابت اختياري واحد على الأقل.

For Example:

$dy/dx = f(x) \quad \rightarrow \quad \int dy = \int f(x) \cdot dx + c$

where c: is essential constant

دائما عدد الثوابت الاختيارية تساوي او اقل من رتبة المعادلة التفاضلية.

2. Special Solution: it is a solution obtained from general solution by finding the values of essential constants. it depended on:

الحل الخاص :هو ذلك الحل الذي نحصل عليه من الحل العام بإيجاد قيم الثوابت الاختيارية .وعادة يحتوي على شروط لإيجاد الثوابت الاختيارية:

a- Boundary conditions for static problems

b- Initial conditions for dynamic problems

3. Singular Solution: solution cannot be found from general solution.

الحل الشاذ :هو ذلك الحل الذي لا يمكن الحصول عليه من الحل العام.

4. Complete Solution: it is a solution that by which can get all the solutions of the differential equation.

الحل التام :هو ذلك الحل الذي يمكن الحصول على جميع حلول المعادلة التفاضلية.

Example (1): Prove that $y = a\, e^{-x} + b\, e^{2x}$ is a general solution for the differential equation $\acute{\acute{y}} - \acute{y} - 2y = 0$?

Solve:

$\acute{\acute{y}} - \acute{y} - 2y = 0$

$y = a\, e^{-x} + b\, e^{2x}$

$\acute{y} = -a\, e^{-x} + 2b\, e^{2x}$

$\acute{\acute{y}} = a\, e^{-x} + 4b\, e^{2x}$

sub $y, \acute{y},$ and $\acute{\acute{y}}$ in above D. E:

$\therefore\ a\, e^{-x} + 4b\, e^{2x} + a\, e^{-x} - 2b\, e^{2x} - 2a\, e^{-x} - 2b\, e^{2x} = 0$

$0 = 0 \qquad \therefore ok$

Example (2): Prove that $\mathrm{y} = \mathrm{a}\, \mathrm{e}^{3x} + \mathrm{b}\, \mathrm{e}^{3x}$ is a general solution for the differential equation $\acute{\acute{y}} - 6\acute{y} + 9y = 0$

Solve:

$\acute{y} = 3\mathrm{a}\, \mathrm{e}^{3x} + 3\mathrm{b}\, \mathrm{e}^{3x}$

$\acute{\acute{y}} = 9\mathrm{a}\, \mathrm{e}^{3x} + 9\mathrm{b}\, \mathrm{e}^{3x}$

Sub y, $\acute{y}$ and $\acute{\acute{y}}$ in above D.E:

$9\mathrm{a}\, \mathrm{e}^{3x} + 9\mathrm{b}\, \mathrm{e}^{3x} - 6(3\mathrm{a}\, \mathrm{e}^{3x} + 3\mathrm{b}\, \mathrm{e}^{3x}) + 9(\mathrm{a}\, \mathrm{e}^{3x} + \mathrm{b}\, \mathrm{e}^{3x}) = 0$

$9\mathrm{a}\, \mathrm{e}^{3x} + 9\mathrm{b}\, \mathrm{e}^{3x} - 18\mathrm{a}\, \mathrm{e}^{3x} - 18\mathrm{b}\, \mathrm{e}^{3x} + 9\mathrm{a}\, \mathrm{e}^{3x} + 9\mathrm{b}\, \mathrm{e}^{3x} = 0$

$18\mathrm{a}\, \mathrm{e}^{3x} + 18\mathrm{b}\, \mathrm{e}^{3x} - 18\mathrm{a}\, \mathrm{e}^{3x} - 18\mathrm{b}\, \mathrm{e}^{3x} = 0$

0=0 $\qquad \therefore ok$

Example (3): Prove that $y = a\cos(2x) + b\sin(2x)$ is a general solution for the differential equation $\acute{\acute{y}} + 4y = 0$

Solve:

$$\acute{y} = -2a\sin(2x) + 2b\cos(2x)$$

$$\acute{\acute{y}} = -4a\cos(2x) - 4b\sin(2x)$$

$$\because\ \acute{\acute{y}} + 4y = 0$$

$$\therefore -4a\cos(2x) - 4b\sin(2x) + 4(a\cos(2x) + b\sin(2x)) = 0$$

$$-4a\cos(2x) - 4b\sin(2x) + 4a\cos(2x) + 4b\sin(2x) = 0$$

0=0 $\therefore ok$

Example (4): Find the general and Special solution for the differential equation $\int \frac{d^2y}{dx^2} = \int 6x\,dx$. The initial condition $\acute{y}(0) = 1$, $y(1) = 0$

Solve:

$$\frac{dy}{dx} = 6\frac{x^2}{2} + c_1 \quad \rightarrow \quad \acute{y} = 3x^2 + c_1$$

$$\because\ \acute{y}(0) = 1 \quad \rightarrow \quad 1 = 3(0) + c_1$$

$$c_1 = 1$$

$$\therefore\ \frac{dy}{dx} = 3x^2 + 1$$

$$\int \frac{dy}{dx} = \int 3x^2\,dx + \int 1dx$$

$$y = x^3 + x + c_2$$

$$\because y(1) = 0 \quad \rightarrow \quad 0 = 1 + 1 + c_2 \quad \rightarrow \quad c_2 = -2$$

$$y = x^3 + x - 2$$

0=1+1-2 → 0=0 $\therefore ok$

Example (5): Prove that $y^2 = ax - x\ln x$ **is a general solution for the differential equation** $\frac{dy}{dx} = \frac{(y^2-x)}{2xy}$ for the values of the constant a .

Solve:

$$y^2 = ax - x\ln x \quad \rightarrow y = (ax - x\ln x)^{\frac{1}{2}}$$

$$\acute{y} = \frac{\left(a - \left[x\frac{1}{x} + lnx\right]\right)}{2(ax - x\ln x)^{\frac{1}{2}}} = \frac{(a - 1 - lnx)}{2(ax - x\ln x)^{\frac{1}{2}}}$$

$$\because \frac{dy}{dx} = \frac{(y^2-x)}{2xy} \quad \rightarrow \frac{(a-1-lnx)}{2(ax - x\ln x)^{\frac{1}{2}}} = \frac{(ax-xlnx-x)}{2x(ax - x\ln x)^{\frac{1}{2}}}$$

$$\frac{(a - 1 - lnx)}{2(ax - x\ln x)^{\frac{1}{2}}} = \frac{x(a - lnx - 1)}{2x(ax - x\ln x)^{\frac{1}{2}}}$$

$$\frac{(a-1-lnx)}{2(ax - x\ln x)^{\frac{1}{2}}} = \frac{(a-lnx-1)}{2(ax - x\ln x)^{\frac{1}{2}}} \qquad \therefore ok$$

H.W: Prove that each of the following differential equation has the given general solution for all values of the constant a and b:

1) $y = 2x^2 \quad , x\acute{y} = 2y$
2) $y = ax + be^x \quad , (x-1)\acute{\acute{y}} - x\acute{y} + y = 0$