A Visual Introduction to 3D Calculus & Functions of two or more independent variables

مقدمة بصرية لحساب التفاضل والتكامل ثلاثي الأبعاد والدوال بمتغيرين مستقلين أو أكثر

Definitions and Theorems: **تعاريف ونظريات**

- In the three-dimensional coordinate system , point are represented by ordered triples (x, y, z). For example, the origin is (0,0,0).

في نظام الإحداثيات ثلاثي الأبعاد، يتم تمثيل النقاط بواسطة ثلاثيات مرتبة (x, y, z). على سبيل المثال، الأصل هو (0,0,0).

- The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

المسافة بين النقاط (x_1, y_1, z_1) , (x_2, y_2, z_2) تحسب وفق الصيغة اعلاه.

- The sphere with center (x_0, y_0, z_0) and radius **r** is the set of all points (x, y, z) such that the distance between (x, y, z) and (x_0, y_0, z_0) is **r**. That is,

$$d = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = r$$

الكرة ذات المركز (x_0, y_0, z_0) ونصف قطرها r هي مجموعة جميع النقاط (x, y, z) بحيث تكون المسافة بين (x, y, z) و (x_0, y_0, z_0) هي r.

This simplifies to the **<u>equation of the sphere</u>**,

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

- The midpoint between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

يتم تحديد نقطة المنتصف بين النقاط (x_1, y_1, z_1) و (x_2, y_2, z_2) بالصيغة اعلاه.

- If z= f(x,y) is function of two variables, than x and y are called the independent variables and z is the dependent variable.

اذا كانت z دالة لمتغيرين فان x & y تسمى متغيرات مستقلة و z تمثل متغير معتمد.

- Let D be a set of ordered pairs of real number. If to each ordered pair (x,y) in D there corresponds a unique real number z= f(x,y), then f is called a function of x and y. The set D is the **<u>domain</u>** of f ,and the corresponding set of values f(x,y) is the **<u>range</u>** of f.

ليكن D مجموعة من الأزواج المرتبة للأعداد الحقيقية. إذا كان لكل زوج مرتب (x,y) في D عدد حقيقي فريد z= f(x,y) ، فإن f تسمى دالة لـ x و y. المجموعة D هي مجال f، والمجموعة المقابلة من القيم $f(x, y)$ هي مدى f.

- The **graph** of a function of two variables z= f(x,y) consists of all points (x,y,z) such that z= f(x,y)

يتكون رسم دالة ذات متغيرين z= f(x,y) من جميع النقاط (x,y,z) بحيث z= f(x,y)

- The **trace** is the intersection of a surface with a plane.

الأثر هو تقاطع السطح مع المستوى.

Example (1): Find the distance between the points (2, -1 , 3) and (1, 0, -2).

Solve: Using the distance formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
$$d = \sqrt{(1-2)^2 + (0+1)^2 + (-2-3)^2} = \sqrt{1+1+25}$$
$$d = \sqrt{27} = 5.196$$

Example (2): Find the distance between the points (1, -2, 4) and (6, -2, -2).

Solve: Using the distance formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
$$d = \sqrt{(6-1)^2 + (-2+2)^2 + (-2-4)^2} = \sqrt{25+0+36}$$
$$d = \sqrt{61} = 7.81$$

Example (3): Find the equation of the sphere with center (0, 2, 5) and radius 2.

Solve: The equation of the sphere is:

$$\boldsymbol{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2}$$
$$(x-0)^2 + (y-2)^2 + (z-5)^2 = (2)^2$$
$$x^2 + (y-2)^2 + (z-5)^2 = 4$$

Example (4): Find the equation of the sphere having (4,-2, 3) & (0, 4, -3) as endpoints as diameter.

Solve: The center of the sphere is the midpoint between the points (4,-2, 3) & (0, 4, -3) is given by the formula:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right) = \left(\frac{4+0}{2}, \frac{-2+4}{2}, \frac{3-3}{2}\right) = (2,1,0)$$

By the distance formula, the radius is:

$$\boldsymbol{r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}}$$
$$r = \sqrt{(0-2)^2 + (4-1)^2 + (-3-0)^2} = \sqrt{4+9+9} = \sqrt{22}$$

The equation of the sphere is: $\boldsymbol{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2}$

$$\boldsymbol{(x-2)^2 + (y-1)^2 + (z-0)^2 = \left(\sqrt{22}\right)^2}$$
$$\boldsymbol{(x-2)^2 + (y-1)^2 + z^2 = 22}$$

Example (5): Find the midpoint of the line segment joining the points (4, 0, -6) and (8, 8, 20).
Solve: The midpoint between the points (4, 0, -6) and (8, 8, 20) is given by the formula:

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)=\left(\frac{4+8}{2},\frac{0+8}{2},\frac{-6+20}{2}\right)=(6,4,7)$$

Example (6): Calculate

1- $f(1,3)$ if $f(x,y)=lny+e^{x+y}$
2- $g(\pi,0)$ if $g(x,y)=3cos(x+y)+sin(x-y)$
3- z(2,1) if $z(x,y)=x^2+xy$

Solve:

1- $f(1,3)=ln3+e^{1+3}=ln3+e^4=1.0986+54.5982=55.6968$
2- $g(\pi,0)=3cos(\pi+0)+sin(\pi-0)$
$=3cos(\pi)+sin(\pi)=3(-1)+0=-3$
3- $z(2,1)=2^2+(2)(1)=4+2=6$

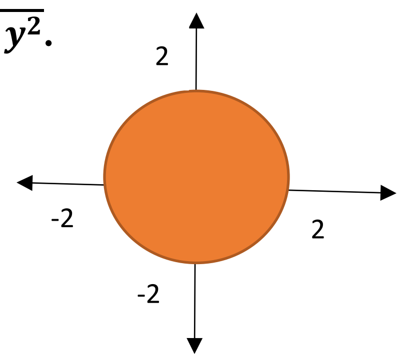
Example (7): Find the domain of the function $f(x,y)=\sqrt{4-x^2-y^2}$.
Solve: يجب أن يكون التعبير داخل الجذر موجب

$4-x^2-y^2\geq 0$

$x^2+y^2\leq 4$

$D=\{(x,y):x^2+y^2\leq 4\}$

The domain is inside a circle of radius 2 and center (0,0,0)



Example (8): Find the domain of the function $f(x,y)=ln(9-x^2-9y^2)$.
Solve:

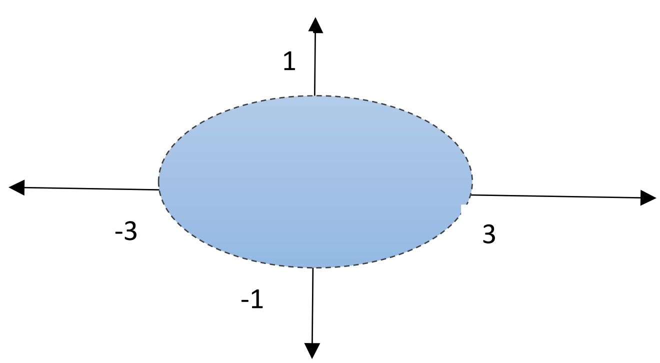
$9-x^2-9y^2>0$

$9>x^2+9y^2 \qquad \div 9$

$1>\frac{x^2}{9}+y^2$

$D=\left\{(x,y):\frac{x^2}{9}+y^2<1\right\}$

The domain is inside the ellipse.



Example (9): Find the domain of the function $\boldsymbol{f(x,y)=\sqrt{x^2+y^2-4}+\sqrt{y-x^2}}$.

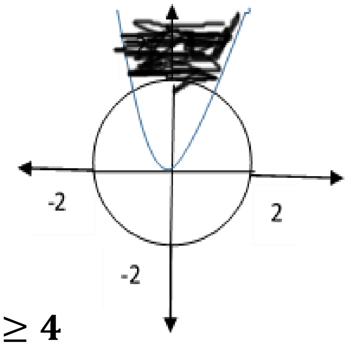
Solve:

1) $\boldsymbol{x^2+y^2-4\geq 0}$

$\boldsymbol{x^2+y^2\geq 4}$ The graph is a circle of radius 2 and center (0,0,0).

2) $\boldsymbol{y-x^2\geq 0}$

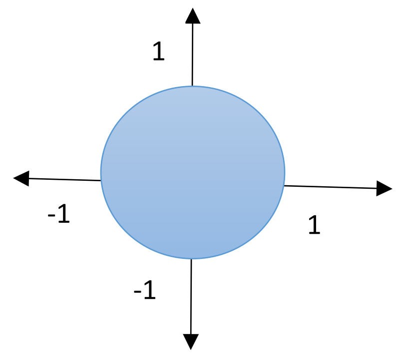
$\boldsymbol{y\geq x^2}$ The graph is a parabola



The domain is inside the parabola $\boldsymbol{y\geq x^2}$ and outside the circle $\boldsymbol{x^2+y^2\geq 4}$

Example (10): Find the domain of the function $\boldsymbol{f(x,y,z)=\sqrt{1-x^2-y^2-z^2}}$.

Solve:



$\boldsymbol{1-x^2-y^2-z^2\geq 0}$

$\boldsymbol{x^2+y^2+z^2\leq 1}$

The domain is inside a sphere of radius 1 and center (0,0,0).