CONSTRAINED EXTREMAL PROBLEMS

The optimization problems having continuous objective function and equality or inequality type constraints are called constrained extremal problems. The solution of such problems, having differentiable objective function and equality type constraints can be obtained by a number of methods, but the most common is the Lagrange multipliers method.

Problem with one Equality Constraint

The use of Lagrange function can best be understood with the help of an example. Let us consider a simple two-variable problem having a single equality type constraint.

$$\begin{aligned} \text{Maximize or minimize } Z &= f(x_1, x_2), \\ \text{subject to } g(x_1, x_2) &= b, \\ x_1, x_2 &\geq 0, \end{aligned}$$

where the objective function as well as the constraint are differentiable w.r.t. x_1 and x_2 and $f(x_1, x_2)$ or $g(x_1, x_2)$ or both are non-linear. The constraint function can be replaced by another differentiable function $h(x_1, x_2)$ such that

$$h(x_1, x_2) = g(x_1, x_2) - b = 0.$$

The problem, then, reduces to

$$\begin{aligned} \text{maximize or minimize } z &= f(x_1, x_2), \\ \text{subject to } h(x_1, x_2) &= 0, \\ x_1, x_2 &\geq 0. \end{aligned}$$

The Lagrangean function can now be formulated as

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda h(x_1, x_2),$$

where λ is the Lagrange multiplier.

The necessary conditions for the maximum or minimum of $f(x_1, x_2)$, subject to the constraint $h(x_1, x_2) = 0$, can be obtained as

$$\begin{aligned} \frac{\partial \mathrm{L}}{\partial x_1} &= 0, \\ \frac{\partial \mathrm{L}}{\partial x_2} &= 0, \\ \frac{\partial \mathrm{L}}{\partial \lambda} &= 0, \\ \mathrm{L} &= \mathrm{L}(x_1, x_2, \lambda). \end{aligned}$$

where
If $f = f(x_1, x_2)$ and $h = h(x_1, x_2)$, the above three necessary conditions for optimization are given by

$$\begin{aligned} &\frac{\partial \mathrm{L}}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0 \quad \text{or} \quad \frac{\partial f}{\partial x_1} = \lambda \frac{\partial h}{\partial x_1}, \\ &\frac{\partial \mathrm{L}}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0 \quad \text{or} \quad \frac{\partial f}{\partial x_2} = \lambda \frac{\partial h}{\partial x_2}, \\ &\frac{\partial \mathrm{L}}{\partial \lambda} = 0 - h = 0 \quad \text{or} \quad -h = 0. \end{aligned}$$

The necessary conditions for optima of $f(x_1, x_2)$, subject to $h(x_1, x_2) = 0$, are thus given by

$$\begin{aligned} f_1 &= \lambda h_1, \\ f_2 &= \lambda h_2, \\ -h &= 0. \end{aligned}$$

and
These necessary conditions are also the sufficient conditions for a maximum if the objective function is concave and for a minimum if the objective function is convex.