Necessary and Sufficient Conditions for a General NLPP

A general NLPP having n variables and m constraints $(n \geq m)$, can be expressed as

$$\begin{aligned} &\text{maximize or minimize} && = f(\mathrm{X}), \mathrm{X} = (x_1, x_2, \dots, x_n), \\ &\text{subject to} \quad g^i(\mathrm{X}) && = b_i, \\ &\mathrm{X} && \geq 0. \end{aligned}$$

The constraint can also be written as

$$h^i(\mathrm{X}) = g^i(\mathrm{X}) - b_i = 0, i = 1,2, \dots, m.$$

By introducing the Lagrange multipliers, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$, the Lagrange function is formed as

$$\mathrm{L}(\mathrm{X}, \lambda) = f(\mathrm{X}) - \sum_{i=1}^{m} \lambda_i h^i(\mathrm{X})$$

Assuming that all the functions L, f and h^i are differentiable partially w.r.t. $x_1, x_2, \dots, x_n$ and $\lambda_1, \lambda_2, \dots, \lambda_m$, the necessary conditions for the objective function to be a maximum or a minimum are

$$\frac{\partial \mathrm{L}}{\partial x_j} = \frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h^i}{\partial x_j} = 0 \quad \text{or} \quad \frac{\partial f}{\partial x_j} = \sum_{i=1}^{m} \lambda_i \frac{\partial h^i}{\partial x_j},$$

$$\text{and} \quad \frac{\partial \mathrm{L}}{\partial \lambda_i} = 0 - h^i = 0 \quad \text{or} \quad -h^i = 0, i = 1,2, \dots, m; j = 1,2, \dots, n.$$

These $(m + n)$ necessary conditions also become the sufficient conditions for a maximum if the objective function is concave and for a minimum if the objective function is convex.

When concavity (convexity) is not known

for an n-variable non-linear programming problem having one equality type constraint, the necessary conditions for a stationary point to be a maximum or minimum are

$$\frac{\partial \mathrm{L}}{\partial x_j} = \frac{\partial f}{\partial x_j} - \lambda \frac{\partial h}{\partial x_j} = 0,\ j = 1,2,\dots,n,$$
$$\frac{\partial \mathrm{L}}{\partial \lambda} = -h(\mathrm{X}) = 0.$$

From the first condition, $\lambda = \frac{\partial f}{\partial x_j} / \frac{\partial h}{\partial x_j}$, for $j = 1,2,\dots,n$.

EXAMPLE

Solve the NLPP:

$$\begin{array}{lll} \text{Maximize} & Z & = 4x_1 - x_1^2 + 8x_2 - x_2{}^2, \\ \text{subject to} & x_1 + x_2 & = 2, \\ & x_1, x_2 & \geq 0. \end{array}$$

Solution

The objective function as well as the constraint are differentiable w.r.t. x_1 and x_2. The constraint can be replaced by another differentiable function such as

$$x_1 + x_2 - 2 = 0$$

The Lagrangean function can be written as

$$\mathrm{L}(x_1, x_2, \lambda) = 4x_1 - x_1^2 + 8x_2 - x_2^2 - \lambda(x_1 + x_2 - 2)$$

The necessary conditions for a maxima or minima of the objective function are and

$$\frac{\partial \mathrm{L}}{\partial x_1} = 4 - 2x_1 - \lambda = 0,$$
$$\frac{\partial \mathrm{L}}{\partial x_2} = 8 - 2x_2 - \lambda = 0,$$

$$\frac{\partial \mathrm{L}}{\partial \lambda} = -(x_1 + x_2 - 2) = 0.$$