D) Integration of function contaning roots

It is done by imposing the quantity that contain a root with any symbol and completing the integration as follows:-

Ex

1. $$\int \frac{1}{1+\sqrt{x}}\,dx$$

let $y=\sqrt{x} \Rightarrow y^2 = x \Rightarrow dx = 2y\,dy$

$$\therefore \int \frac{1}{1+\sqrt{x}}\,dx = \int \frac{1}{1+y}\,2y\,dy$$

$$= 2\int \frac{y}{1+y}\,dy$$

Since the order of the numerator is equal to the order of the denominator, we use long division as follows:

$$\begin{array}{r|l} & \quad 1 \\ 1+y & \quad y \\ & \mp y \mp 1 \\ \hline & \quad -1 \end{array}$$

$$\therefore \int \frac{1}{1+\sqrt{x}}\,dx = 2\int \frac{y}{1+y}\,dy$$

$$= 2\left[\int 1\,dy - \int \frac{1}{1+y}\,dy\right]$$

$$= 2(y - \ln|1+y|) + c$$

$$= 2\sqrt{x} - 2\ln|1+\sqrt{x}| + c$$

2. $\int \frac{1+\sqrt{x}}{1-\sqrt{x}}\,dx$

let $y=\sqrt{x} \Rightarrow y^2=x \Rightarrow dx=2y\,dy$

$\therefore \int \frac{1+\sqrt{x}}{1-\sqrt{x}}\,dx = \int \frac{1+y}{1-y}\,2y\,dy$

$= \int \frac{2y+2y^2}{1-y}\,dy$

$$\begin{array}{r} y+2 \\ 1-y \,\overline{)\, y^2+y} \\ -y^2+y \\ \hline 2y \\ -2y+2 \\ \hline 2 \end{array}$$

$= 2\left[\int (y+2)\,dy + \int \frac{2}{1-y}\,dy\right]$

$= 2\left(\frac{y^2}{2} + 2y + 2\ln|1-y|\right)$

$= y^2 + 4y + 4\ln|1-y| + c$

$= x + 4\sqrt{x} + 4\ln|1-\sqrt{x}| + c$

H.W.

1. $\int \frac{1}{1+\sqrt[4]{x}}\,dx$

let $\sqrt[4]{x}=y \Rightarrow y^4=x \Rightarrow dx=4y^3\,dy$

$\therefore \int \frac{1}{1+\sqrt[4]{x}}\,dx = \int \frac{1}{1+y}\,4y^3\,dy$

$$\begin{array}{r} y^2+y+ \\ 1+y \,\overline{)\, y^3} \\ \mp y^3 \mp y^2 \\ \hline -y^2 \\ \mp y^2 \pm y \\ \hline y \\ \mp y \mp 1 \\ \hline -1 \end{array}$$

$= 4\int \frac{y^3}{1+y}\,dy$

$= 4\left[\int (y^2+y+1)\,dy - \int \frac{1}{1+y}\,dy\right]$

$$= 4\frac{y^3}{3} + 4\frac{y^2}{2} + 4y - \ln|1+y| + c$$

$$= \frac{4}{3}(\sqrt[4]{x})^3 + 2(\sqrt[4]{x})^2 - 4(\sqrt[4]{x}) - \ln|1+\sqrt[4]{x}| + c$$

$$= \frac{4}{3}(x)^{\frac{3}{4}} + 2(x)^{\frac{1}{2}} - 4(x)^{\frac{1}{4}} - \ln|1+(x)^{\frac{1}{4}}| + c$$

2. $\int 2\sqrt{2x-1}\, dx$

let $y = \sqrt{2x-1} \Rightarrow y^2 = 2x-1 \Rightarrow x = \frac{y^2-1}{2}$

$$dx = \frac{2(2y) - (y^2-1)(0)}{4} = y\,dy$$

$$\therefore \int 2\sqrt{2x-1}\, dx = \int 2y \cdot y\,dy$$

$$= 2\int y^2 dy$$

$$= 2\frac{y^3}{3} + c$$

$$= \frac{2}{3}(\sqrt{2x-1})^3 + c$$

$$= \frac{2}{3}(2x-1)^{\frac{3}{2}} + c$$