University of Mosul
College of Computer Science and Mathematics
Department of Statistics and Informatics
Stochastic Processes (1)
Level 4

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Lecture One of the Stochastic Processes (1) course.

Vocabulary for Subjects of Stochastic Processes (1)

Section One: Introduction

- Basic review of probability.
- Probability generating functions of random variables.
- Probability generating function of the sum of a fixed number of random variables.
- Probability generating function of the sum of a random number of random variables.
- Probability generating function of a bivariate distribution.

Section Two: Stochastic Processes

- Introduction to stochastic processes and their definitions.
- Specification of stochastic processes.
- Classification of stochastic processes.
- Introduction to Markov chains and their definitions.
- The initial distribution and transition matrix.
- Higher-order transition probabilities (Chapman–Kolmogorov equation).
- Determination of higher-order transition probabilities.
- Applications and examples.

Definitions

Random Variable. Consider an experiment with a probability measure $P(\cdot)$ defined on a sample space S and a function that assigns a real number to each outcome in the sample space of the experiment. This function is called a *random variable*.

Probability Mass Function (p.m.f). A probability mass function (p.m.f) is a function that gives the probability that a discrete random variable is exactly equal to some value. The probability mass function is often the primary means of defining a discrete probability distribution.

Let X be a discrete random variable with range $S = \{x_1, x_2, \ldots\}$ (a countable set). Then the function

$$P_X(x) = P\{X = x\}, \quad x \in S,$$

is called the probability mass function (p.m.f) of X.

Then:

$$\sum_x P_X(x) = 1, \qquad 0 \leq P_X(x) \leq 1 \text{ for all } x.$$

For any set $B \subset S$,

$$P[X \in B] = \sum_{x \in B} P_X(x).$$

The distribution function is $F_X(x) = P\{X \leq x\}$, and the mean is

$$E(X) = \mu_X = \sum_{x \in S} x\, P_X(x).$$

Generating Function. Let $\{a_0, a_1, a_2, \ldots\}$ be a sequence of real numbers. Using a variable S, define

$$A(S) = a_0 + a_1 S + a_2 S^2 + \cdots = \sum_{k=0}^{\infty} a_k S^k. \tag{1.1}$$

Then $A(S)$ is called the generating function of the sequence $\{a_k\}_{k=0}^{\infty}$. By differentiating k times, putting $(S = 0)$, and dividing by $(k!)$, we get

$$a_k = \frac{1}{k!}\left[\frac{\partial^k A(S)}{\partial S^k}\right]_{S=0}. \tag{1.2}$$

Probability Generating Function

Suppose that X is a random variable which assumes non-negative integers $0, 1, 2, \ldots$, and that

$$P\{X = k\} = p_k, \qquad k = 0, 1, 2, \ldots, \quad \sum_{k=0}^{\infty} p_k = 1. \tag{1.3}$$

If we take a_k to be the probability p_k, then the corresponding function $P(S)$ of the sequence of probabilities $\{p_k\}$ is known as the probability generating function (p.g.f) of the random variable X:

$$P(S) = \sum_{k=0}^{\infty} p_k S^k = \sum_{k=0}^{\infty} P_r\{X = k\} S^k = E(S^k). \tag{1.4}$$

The series $P(S)$ converges for at least $-1 \leq S \leq 1$, and clearly $P(1) = 1$.

The first two derivatives of $P(S)$ are:

$$P'(S) = \sum_{k=1}^{\infty} k\, p_k\, S^{k-1}, \qquad P''(S) = \sum_{k=2}^{\infty} k(k-1)\, p_k\, S^{k-2}.$$

The Mean and Variance of X

By putting $(S = 1)$ in the first derivative of $P(S)$, we get the expectation of X, i.e.

$$E(X) = P'(1) = \sum_{k=1}^{\infty} k\, p_k. \tag{1.5}$$

Also,

$$P''(1) = \sum_{k=2}^{\infty} k(k-1)\, p_k = E\big[X(X-1)\big] = E(X^2) - E(X).$$

Then

$$E(X^2) = P''(1) + E(X) = P''(1) + P'(1).$$

Hence the variance of X is:

$$\text{var}(X) = E(X^2) - [E(X)]^2 = P''(1) + P'(1) - \big(P'(1)\big)^2. \tag{1.6}$$

More generally, the k^{th} factorial moment of X is:

$$E\big[X(X-1)(X-2)\cdots(X-k+1)\big] = \left[\frac{\partial^k P(S)}{\partial S^k}\right]_{S=1}, \qquad k = 1, 2, \ldots$$

Note that: $P(e^S)$ is the moment generating function.