Lecture 10: Higher Transition Probability

Higher Transition Probability:

Let p_{ij} denoted the transition probability from state i to state j in one-step transition:

$$p_{ij} = Pr\{x_{n+1} = j \mid x_n = i\}$$

And the m-Step transition probability is given by:

$$p_{ij}^{(m)} = Pr\{x_{n+m} = j \mid x_n = i\},$$

is the transition probability from state i to state j in m-step transition, where:

$$p_{ij}^{(2)} = Pr\{x_{n+2} = j \mid x_n = i\}$$

is a two-step transition probability, means that the state j can be reached from state i in two steps through some intermediate state r.

Chapman-Kolmogorov Equation:

Consider a fixed value of r, where:

$$Pr\{x_{n+2} = j,\, x_{n+1} = r \mid x_n = i\}$$

$$= Pr\{x_{n+2} = j \mid x_{n+1} = r, x_n = i\}\, Pr\{x_{n+1} = r \mid x_n = i\},$$

(by Markov property)

$$= Pr\{x_{n+2} = j \mid x_{n+1} = r\} Pr\{x_{n+1} = r \mid x_n = i\}$$

$$= p_{rj}^{(1)} \cdot p_{ir}^{(1)} = p_{ir} \cdot p_{rj}.$$

Since these intermediate states $r = 1, 2, 3, \ldots$ are mutually exclusive states, we have:

$$p_{ij}^{(2)} = Pr\{x_{n+2} = j \mid x_n = i\} = \sum_r Pr\{x_{n+2} = j, x_{n+1} = r \mid x_n = i\}$$

$$= \sum_r p_{ir} \cdot p_{rj}.$$

(Summing over all the intermediate states)

By mathematical induction, we have:

$$p_{ij}^{(m+1)} = Pr\{x_{n+m+1} = j \mid x_n = i\}$$

$$= \sum_r Pr\{x_{n+m+1} = j, x_{n+m} = r \mid x_n = i\}$$

$$= \sum_r p_{ir}^{(1)} \cdot p_{rj}^{(m)} = \sum_r p_{ir}^{(m)} \cdot p_{rj}^{(1)}.$$

In general:

$$p_{ij}^{(m+n)} = \sum_r p_{ir}^{(n)} \cdot p_{rj}^{(m)} = \sum_r p_{ir}^{(m)} \cdot p_{rj}^{(n)}.$$

This equation is special case of Chapman-Kolmogorov equation which is satisfied by the transition probabilities of Markov Chain.

Since:

$$P = [p_{ij}] = \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots & p_{0n} \\ p_{10} & p_{11} & p_{12} & \cdots & p_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{n0} & p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix},$$

Then: $P^{(m)} = [p_{ij}^{(m)}]$, m-step.

And:

$$P^{(m+1)} = P^m \cdot P = P \cdot P^m$$

In general: $P^{(m+n)} = P^m \cdot P^n = P^n \cdot P^m$.

For example:

$$P^{(2)} = P^2 = P \cdot P$$

$$P^{(3)} = P^3 = P^2 \cdot P = P \cdot P^2$$

$$P^{(5)} = P^5 = P^4 \cdot P = P^3 \cdot P^2 = P \cdot P^4$$

Theorem:

Let P be the transition matrix of a Markov Chain, then the m-step transition matrix is equal to the mth power of P, i.e.:

$$P^{(m)} = P^m.$$

Example :

Consider a M.C. with transition matrix:

$$P = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad S = \{1, 2\},$$

then find:

1) 2-step transition matrix.
2) 4-step transition matrix.
3) 5-step transition matrix.
4) $p_{12}^{(2)}, p_{22}^{(4)}, p_{12}^{(1)}$.

Solution:

$$P^{(2)} = P^2 = P \cdot P = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

$$P^{(4)} = P^4 = P^2 \cdot P^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \\ \frac{5}{16} & \frac{11}{16} \end{bmatrix}$$

$$P^{(5)} = P^5 = P^4 \cdot P = \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \\ \frac{5}{16} & \frac{11}{16} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{16} & \frac{11}{16} \\ \frac{11}{32} & \frac{21}{32} \end{bmatrix}$$

$$p_{12}^{(2)} = \frac{1}{2}, \qquad p_{22}^{(4)} = \frac{11}{16}, \qquad p_{12}^{(1)} = 1$$

Example :

Three students A, B, and C are using only one calculator. The student A gives the calculator to B or C after he finishing his work, but B gives the calculator to C always, so student C flips two coins, if two heads occur, then he gives the calculator to A, otherwise he gives the calculator to B.

1. Write the transition matrix for this example.
2. When the calculator with C, what the probability that he will gives to B.
3. If the calculator with A, what the probability it will be with B after two-step.

Solution:

The state space is $S = \{A, B, C\}$.

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix}$$

$$Pr\{x_{n+1} = B \mid x_n = C\} = p_{CB} = \frac{3}{4}$$

$$p_{AB}^{(2)} = Pr\{x_{n+2} = B \mid x_n = A\} = Pr\{x_2 = B \mid x_0 = A\}$$

$$P^{(2)} = P^2 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & \frac{3}{8} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & \frac{1}{8} & \frac{7}{8} \end{bmatrix}$$

$$p_{AB}^{(2)} = \frac{3}{8}.$$