Lecture 11: Initial Distribution and Probability Distribution

Initial Distribution and Probability Distribution:

Let the stochastic vector:

$$p^{(0)} = \left[p_0^{(0)} \; p_1^{(0)} \; p_2^{(0)} \; \ldots \right],$$

with the state space

$$S = \{0, 1, 2, \ldots\},$$

denote the initial probability distribution (the distribution when the process begins), where:

$$P_r\{x_0 = i\} = p_i^{(0)}, \qquad i = 0, 1, 2, \ldots$$

Then, after the first n steps we have:

$$p^{(n)} = \left[p_0^{(n)} \; p_1^{(n)} \; p_2^{(n)} \; \ldots \right]$$

denoting the n^{th} step probability distribution, where:

$$P_r\{x_n = i\} = p_i^{(n)}, \qquad i = 0, 1, 2, \ldots$$

and

$$p^{(n)} = p^{(n-1)} P.$$

That is,

$$p^{(n)} = \left[p_0^{(n-1)} \; p_1^{(n-1)} \; p_2^{(n-1)} \; \ldots \right] \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Theorem :

Let P be the transition matrix of a M.C., and let $p^{(0)}$ be the initial probability distribution of the process, then:

$$p^{(n)} = p^{(0)} P^n,$$

which is the probability distribution after n-steps.

Proof:

Since

$$p^{(n)} = p^{(n-1)}P,$$

then:

$$p^{(1)} = p^{(0)}P,$$

$$p^{(2)} = p^{(1)}P = p^{(0)}P \cdot P = p^{(0)}P^2,$$

$$p^{(3)} = p^{(2)}P = p^{(0)}P^2 \cdot P = p^{(0)}P^3,$$

and in general:

$$p^{(n)} = p^{(0)}P^n.$$

Example :

If we have a M.C. with state space $S = \{A, B, C\}$ and initial distribution:

$$p^{(0)} = [0 \;\; 0 \;\; 1],$$

and transition matrix:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix},$$

find: $p^{(1)}, p_A^{(2)}, p_C^{(2)}, P_r\{x_3 = B\}$.

Solution:

$$p^{(1)} = p^{(0)}P = [0 \; 0 \; 1] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2} & \dfrac{1}{2} & 0 \end{bmatrix}.$$

Next:

$$p^{(2)} = p^{(1)}P = \begin{bmatrix} \dfrac{1}{2} & \dfrac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = [0 \;\; \tfrac{1}{2} \;\; \tfrac{1}{2}].$$

Thus:

$$p_A^{(2)} = 0, \qquad p_C^{(2)} = \frac{1}{2}.$$

Now:

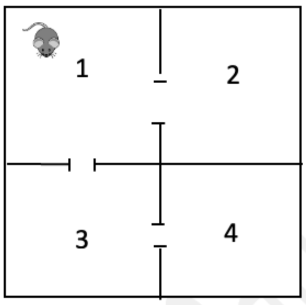
$$p^{(3)} = p^{(2)}P = [0 \;\; \tfrac{1}{2} \;\; \tfrac{1}{2}] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{4} & \dfrac{1}{4} & \dfrac{1}{2} \end{bmatrix}.$$

Thus:

$$P_r\{x_3 = B\} = p_B^{(3)} = \frac{1}{4}.$$

Example :

Mouse found in one of four rooms shown in the figure below. It moving between rooms a day at a random and not remain in one place. Suppose that $x_n$is the room operated by mouse.



Find:

1. The transition matrix and state space. 2. $P_r\{x_1 = 1 \mid x_0 = 2\}$. 3. $P_r\{x_2 = 1 \mid x_0 = 4\}$. 4. If starting in room 1, find the distribution after two days.

Solution:

State space:

$$S = \{1, 2, 3, 4\}.$$

Transition matrix:

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

2)

$$P_r\{x_1 = 1 \mid x_0 = 2\} = p_{21} = 1.$$

3)

$$P_r\{x_2 = 1 \mid x_0 = 4\} = p_{41}^{(2)}.$$

Compute:

$$P^2 = P \cdot P = \begin{bmatrix} \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

Thus:

$$p_{41}^{(2)} = \frac{1}{2}.$$

4) Since the mouse is seen in the first room, then the initial distribution will be in the following

$$p^{(0)} = [1\ 0\ 0\ 0].$$

Then the probability distribution after two-steps (two days) is:

$$p^{(2)} = p^{(0)} P^2 = [\tfrac{3}{4}\ \ 0\ \ 0\ \ \tfrac{1}{4}].$$

Example : H.W

If we have a M.C. with state space $S = \{1, 2, 3\}$, and the following transition matrix:

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad p^{(0)} = \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{3} \end{bmatrix}.$$

Find:

$$p^{(2)},\ p_{13}^{(2)},\ p_3^{(2)},\ P_r\{x_3 = 2\},\ p_1^{(4)}.$$

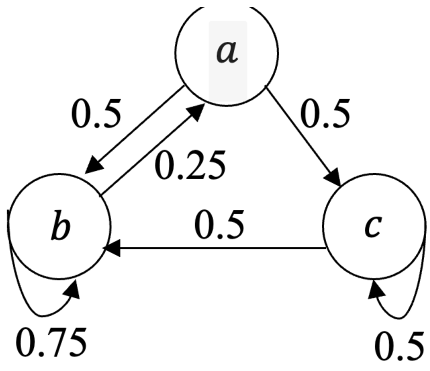
Transition Diagram and Transition Tree:

1) Transition Diagram

The transition probability of Markov Chain can be represented by a diagram call (Transition Diagram), where a probability p_{ij} is denoted by an arrow from state i to state j, for example consider the following transition matrix:

$$P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.25 & 0.75 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}, \quad S = \{a, b, c\},$$

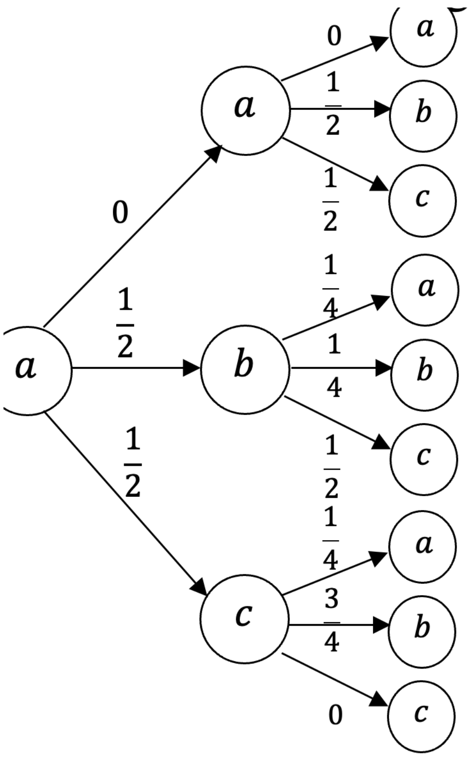
Then the transition diagram for this process is:



2) Transition Tree

We can use the transition tree to find the n-step transition from state i to state j, for example consider the following transition matrix:

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix}, \quad S = \{a, b, c\},$$



the transition tree yields:

1- $Pr\{x_2 = b \mid x_0 = a\}$:

$$Pr\{x_2 = b \mid x_0 = a\} = p_{ab}^{(2)} = \left(0 \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{4}\right) + \left(\frac{1}{2} \times \frac{3}{4}\right)$$

$$= 0 + \frac{1}{8} + \frac{3}{8} = \frac{4}{8} = \frac{1}{2}$$

2- $Pr\{x_2 = c \mid x_0 = a\} = p_{ac}^{(2)} = \left(0 \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times 0\right)$

$$= 0 + \frac{1}{4} + 0 = \frac{1}{4}$$

3- $Pr\{x_2 = c \mid x_0 = b\} = p_{bc}^{(2)} = \left(\frac{1}{4} \times \frac{1}{2}\right) + \left(\frac{1}{4} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times 0\right)$

$$= \frac{1}{8} + \frac{1}{8} + 0 = \frac{1}{4}$$