Lecture Two of the Stochastic Processes (1) course.

Examples

Example Let X be a Poisson variate with probability mass function:

$$p_k \;=\; P_r\{X=k\} \;=\; \frac{e^{-\lambda}\,\lambda^k}{k!}\,, \qquad k=0,1,2,\ldots$$

The p.g.f. of Poisson distribution is:

$$\begin{aligned}
P(S) &= \sum_{k=0}^{\infty} p_k S^k \\
&= \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!}\,S^k \\
&= e^{-\lambda}\sum_{k=0}^{\infty} \frac{(\lambda S)^k}{k!} \\
&= e^{-\lambda}\,e^{\lambda S} \qquad \text{where: } \; e^X = \sum_{k=0}^{\infty}\frac{X^k}{k!} \\
&= e^{\lambda(S-1)}\,, \quad \text{the p.g.f. of Poisson distribution.}
\end{aligned}$$

Then the mean and variance of X is:

$$E(X) = P'(1), \qquad \text{then } \; P'(S) = \lambda e^{\lambda(S-1)}.$$

Put $S=1$, we have:

$$P'(1) = \lambda e^{\lambda(S-1)}\big|_{S=1} = \lambda e^0 = \lambda\,, \quad \text{the mean of } X.$$

$$\text{var}(X) = P''(1) + P'(1) - [P'(1)]^2, \qquad P''(S) = \lambda^2 e^{\lambda(S-1)} \;\Rightarrow\; P''(1) = \lambda^2.$$

Then:

$$\text{var}(X) = \lambda^2 + \lambda - \lambda^2 = \lambda\,, \quad \text{the variance of } X.$$

Example: Let X be a random variable with Geometric distribution p_k where:

$$p_k = P_r\{X=k\} = q^k p\,, \qquad k=0,1,2,\ldots$$

Find the p.g.f of X and the mean and variance of it.

Solution: The p.g.f. of Geometric distribution is:

$$\begin{aligned} P(S) &= \sum_{k=0}^{\infty} p_k S^k = \sum_{k=0}^{\infty} q^k p\, S^k \\ &= p \sum_{k=0}^{\infty} (qS)^k \quad \text{since } \sum_{k=0}^{\infty} a^k = \frac{1}{1-a} \\ &= \frac{p}{1-qS} \ , \end{aligned}$$

the p.g.f. of Geometric distribution. Then the mean and variance of X is:

$$E(X) = P'(1),$$
$$P'(S) = -p(1-qS)^{-2}(-q) = \frac{pq}{(1-qS)^2} \ \Rightarrow \ E(X) = \frac{pq}{(1-qS)^2}\bigg|_{S=1} = \frac{pq}{p^2} = \frac{q}{p} \ .$$

the mean of X.

$$\text{var}(X) = P''(1) + P'(1) - [P'(1)]^2,$$
$$P''(S) = -2pq(1-qS)^{-3}(-q) = \frac{2pq^2}{(1-qS)^3} \ \Rightarrow \ P''(1) = \frac{2pq^2}{p^3} = \frac{2q^2}{p^2}.$$

Then:

$$\text{var}(X) = \frac{2q^2}{p^2} + \frac{q}{p} - \left(\frac{q}{p}\right)^2 = \frac{q}{p^2} \ ,$$

the variance of X.

Example: Let X be a r.v. with Binomial distribution with (p.m.f) p_k, where:

$$p_k = P_r\{X = k\} = \binom{n}{k} p^k q^{n-k} \ , \qquad k = 0, 1, 2, \ldots$$

Find the mean and variance of X by using p.g.f.

Solution: The p.g.f. of Binomial distribution is:

$$\begin{aligned} P(S) &= \sum_{k=0}^{\infty} p_k S^k = \sum_{k=0}^{\infty} \binom{n}{k} p^k q^{n-k} S^k \\ &= \sum_{k=0}^{\infty} \binom{n}{k} (pS)^k q^{n-k} \ = \ (pS+q)^n \ , \end{aligned}$$

the p.g.f. of Binomial distribution (by binomial theorem). Then the mean and variance of X is:

$$E(X) = P'(1),$$
$$P'(S) = np(pS+q)^{n-1} \ \Rightarrow \ E(X) = np(pS+q)^{n-1}\Big|_{S=1} = np(p+q)^{n-1} = np \ ,$$

the mean of X.

$\mathrm{var}(X) = P''(1) + P'(1) - [P'(1)]^2,$
$P''(S) = n(n-1)p^2(pS+q)^{n-2} \;\Rightarrow\; P''(1) = n(n-1)p^2(p+q)^{n-2} = n(n-1)p^2.$

Then:

$$\mathrm{var}(X) = n(n-1)p^2 + np - (np)^2 = n^2p^2 - np^2 + np - n^2p^2 = np(1-p)\;,$$

the variance of X.