Lecture Three of the Stochastic Processes (1) course.

Example:

Let X be a Bernoulli variate p_k where:

$$p_k = P_r\{X = k\} = p^k(1-p)^{1-k}, \qquad k = 0, 1.$$

Find the p.g.f of X, and the mean and variance of it.

Solution: The p.g.f. of Bernoulli distribution is:

$$\begin{aligned}
P(S) &= \sum_{k=0}^{\infty} p_k S^k \\
&= \sum_{k=0}^{1} p^k(1-p)^{1-k} S^k \\
&= \sum_{k=0}^{1} (pS)^k(1-p)^{1-k} \\
&= (pS)^0(1-p)^1 + (pS)^1(1-p)^0 \\
&= (1-p) + pS \quad \text{since } q = 1-p \\
&= q + pS \ , \quad \text{the p.g.f. of Bernoulli distribution.}
\end{aligned}$$

Then the mean and variance of X is:

$$E(X) = P'(1), \qquad P'(S) = p.$$

Thus,

$$E(X) = p|_{S=1} = p \ , \quad \text{the mean of } X.$$

$$\text{var}(X) = P''(1) + P'(1) - [P'(1)]^2.$$

Since $P''(1) = 0$, then:

$$\text{var}(X) = 0 + p - p^2 = p(1-p) = pq \ , \quad \text{the variance of } X.$$

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Example :

Suppose that X is a random variable distributed as Geometric distribution with probability function:

$$p_k = P_r\{X = k\} = q^{k-1}p, \qquad k = 1, 2, \ldots$$

Find the p.g.f of X, and the mean and variance of it.

Solution: The p.g.f. of Geometric distribution is:

$$P(S) = \sum_{k=0}^{\infty} p_k S^k = \sum_{k=1}^{\infty} q^{k-1} p S^k$$
$$= pS \sum_{k=1}^{\infty} (qS)^{k-1}.$$

Let $u = k - 1$, then:

$$P(S) = pS \sum_{u=0}^{\infty} (qS)^u = \frac{pS}{1 - qS}, \quad \text{the p.g.f. of Geometric distribution.}$$

Then the mean and variance of X is:

$$E(X) = P'(1).$$

Differentiate:

$$P'(S) = \frac{(1 - qS)p - pS(-q)}{(1 - qS)^2} = \frac{p - pqS + pqS}{(1 - qS)^2} = \frac{p}{(1 - qS)^2}.$$

Thus,

$$E(X) = \frac{p}{(1 - q)^2}\bigg|_{S=1} = \frac{p}{p^2} = \frac{1}{p}.$$

$$\text{var}(X) = P''(1) + P'(1) - [P'(1)]^2.$$

Now,

$$P''(S) = -2p(1 - qS)^{-3}(-q) = \frac{2pq}{(1 - qS)^3} \Rightarrow P''(1) = \frac{2pq}{p^3} = \frac{2q}{p^2}.$$

Then:

$$\text{var}(X) = \frac{2q}{p^2} + \frac{1}{p} - \left(\frac{1}{p}\right)^2 = \frac{2q + p - 1}{p^2} = \frac{2(1 - p) + p - 1}{p^2} = \frac{q}{p^2}.$$

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Example : (H.W.)

Let X have a Zero-truncated Poisson distribution with zero class missing:

$$p_k = P_r\{X = k\} = \frac{a^k}{k!\,(e^a - 1)}, \qquad k = 1, 2, \ldots$$

Show that: 1. $P(S) = (e^a - 1)^{-1}(e^{aS} - 1)$ 2. Verify that $P(1) = \sum_{k=1}^{\infty} p_k = 1$ 3. $E(X) = \dfrac{ae^a}{e^a - 1}$

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Example:

Let X be a r.v. with p.g.f $P(S)$. Find the p.g.f of the r.v.

$$Y = mX + n,$$

where m and n are integers and $m \neq 0$.

Solution: Let $P_x(S)$ and $P_y(S)$ be the p.g.f of X and Y respectively, then:

$$P_y(S) = E(S^Y) = E(S^{mX+n}) = E(S^n \cdot S^{mX}) = S^n E[(S^m)^X] = S^n P_x(S^m).$$