Lecture 1: Numerical Analysis (2)

Interpolation

To understand **interpolation** (also known as **interpolation** or **insertion**), suppose we are given the following pairs of values:

$(x_0, y_0), (x_1, y_1) \dots, (x_n, y_n)$,where n is a positive integer. In this case, **interpolation** is the process of finding the value of $y = f(x)$ at the point x , where $x_0 < x < x_n$ and $x_i \neq$x for all i=0,1,2,….n. This means that the value of x lies within the range of the given data, but it does not equal any of the given values. Hence, we use the term "interpolation" because the value of x is interpolated between the known values. The main problem, therefore, is how to find the interpolation function $f(x)$ from the given data pairs. There are several methods for estimating the interpolation function $f(x)$, which we will discuss in the upcoming sections.

Linear Interpolation

In **linear interpolation**, the value of y is estimated using x through the following linear relationship:

$$y = B_0 + B_1 x$$

B_1, B_0 The values are estimated based on the given data points. To do so, we select two points (pairs of values) from the given data, such as $(x_a, y_a), (x_b, y_b)$ and with the condition that x lies between these two points, i.e., $x_a < x < x_b$ It is preferred that the values x_a and x_b be as close as possible to the value of x to ensure that the relationship between these two points is as linear as possible. Thus, we do not need to use all the values provided in the problem; we only need to identify two points. Once the two points are identified, we can solve for y using one of the following methods:

Method 1: Equal Slope Method

This method is specific to linear interpolation. Suppose we have a straight line between the two points $(x_a, y_a), (x_b, y_b)$, and the point (x, y)lies on the straight line $\overline{(x_a, y_a),(x_b, y_b)}$. This means that the slope between the points $(x_a, y_a), (x_b, y_b)$ is equal to the slope between the points $(x_a, y_a), (x, y)$, i.e.,

$$slope = \frac{\Delta y}{\Delta x} = \frac{y_b - y_a}{x_b - x_a} = \frac{y - y_a}{x - x_a}$$

From this, we can solve for y.

Thus, **linear interpolation** is an easy and clear method, but it does not provide highly accurate results because the relationship between the two selected points may not be linear. Additionally, using only two points to construct the interpolation function disregards the information available from the other data points.

1- System of Linear Equations Method

This method can be applied to find the interpolation function for any degree, whether linear, quadratic, or even higher-degree functions. As previously mentioned, linear interpolation uses the following linear relationship:

$$y = B_0 + B_1 x$$

After selecting the two points $(x_a, y_a), (x_b, y_b)$, we substitute them into the above equation as follows:

$$y_a = B_0 + B_1 x_a$$
$$y_b = B_0 + B_1 x_b$$

Since the values $(x_a, y_a), (x_b, y_b)$ are already known in the problem, we can see that the system consists of two unknowns and two equations. This system can be converted into a matrix form as follows:

$$\begin{bmatrix} 1 & x_a \\ 1 & x_b \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \end{bmatrix} = \begin{bmatrix} y_a \\ y_b \end{bmatrix}$$

We can solve the system using one of the methods we covered previously in the first semester, such as **Gaussian elimination**, **Gaussian-Jordan method**, **inverse method**, or **triangular factorization**, as well as other numerical methods like the **Jacobi method** or **Gauss-Seidel method**.

After solving, we substitute the values of mmm and B_1, B_0 back into the linear relationship to find the estimated value of y.

Example:

The data below represents the speed of a specific rocket from the moment of launch until the 30th second after the desired launch. Estimating the speed of the rocket at 16 seconds using linear interpolation.

Time in seconds	0	10	15	20	22.5	30
Speed km/min	0	227.04	362.78	517.35	602.97	901.67

Solution:

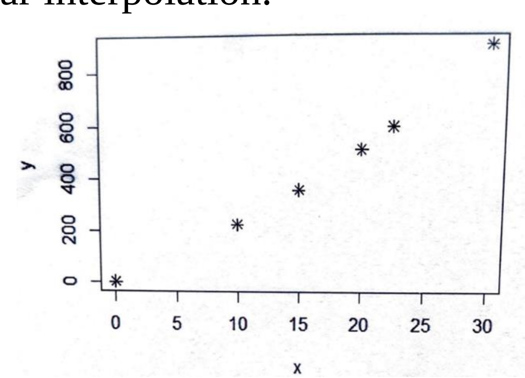
(Equal two slope method):
Clearly, the variable x represents time, while the variable y represents the rocket's speed. In other words, the speed $f(x) = y$ is known for the given time x. Since the task is to estimate the rocket's speed when $x = 16$, the closest two values are $x_a = 15$ and $x_b = 20$, with corresponding speeds $y_a = 362.78$ and $y_b = 517.35$. Substituting into the formula:

$$y = y_a + (x - x_a)\frac{y_b - y_a}{x_b - x_a}$$

$$y = 362.78 + (16 - 15)\frac{517.35 - 362.78}{20 - 15} = 393.694$$

We find:

$y = 393.694$ km/miny
Thus, the rocket's speed at 16 seconds is approximately 393.694 km/min according to the linear interpolation.



System of Linear Equations Method:

To estimate the values B_1, B_0 from the given data, we substitute the points

$$\left(x_a, y_a\right) = \left(x_b, y_b\right) =, \left(15, 362.78\right)\left(20, 517.35\right)$$

into the straight-line equation:

$$y_a = B_0 + B_1 x_a$$
$$362.78 = B_0 + B_1\left(15\right)$$

$$\mathrm{y_b = B_0 + B_1 x_b}$$
$$517.35 = \mathrm{B_0 + B_1}\left(20\right)$$

"Now, it can be observed that the last two equations form a system of linear equations, where we have two equations and two unknowns B_1, B_0. Therefore, we can apply one of the methods for solving a system of linear equations that we discussed in the first chapter. For example, we can use the Gauss-Jordan method to solve the above problem as follows:"

$$B_0 + 15B_1 = 362.78$$
$$B_0 + 20B_1 = 517.35$$

Using the Matrix Method

$$Ax = b$$
$$\begin{bmatrix} 1 & 15 \\ 1 & 20 \end{bmatrix}\begin{bmatrix} B_0 \\ B_1 \end{bmatrix} = \begin{bmatrix} 362.78 \\ 517.35 \end{bmatrix}$$

Now, the augmented matrix is written, and then the necessary transformations are applied to convert the augmented matrix into the identity matrix, as follows:

$$\begin{bmatrix} 1 & 15: & 362.78 \\ 1 & 20: & 517.35 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0: & -100.93 \\ 0 & 1: & 30.914 \end{bmatrix}$$

This means that the equation of the straight line we will apply in the linear interpolation will be as follows:

$$y = -100.93 + 30.914x$$

Note that the validity of the final model is limited to all values of x that satisfy the condition $15 < x < 20$, so when $x = 16$ is:

$$y = -100.93 + 30.914 \times 16 = 393.694$$

This is the same result we reached using equation (1).

Homework:

1- *For the above example data, use linear inter*polation *to estimate the speed of the rocket when x=24 By applying the method of* Equal two slop.

2- For the above example data, use linear interpolation to estimate the speed of rocket when

x =21 by solving a system of linear equations using the inverse matrix method.