Lecture Four of the Stochastic Processes (1) course.

P.G.F. of Sum of Fixed Number of r.v.'s

Let X and Y be two independent non-negative integer-valued random variables with probability distribution given by:

$$P\{X = k\} = a_k, \qquad P\{Y = j\} = b_j.$$

The sum $Z = X + Y$ is r.v., then the event $[Z = r]$ can happen in the following mutually exclusive ways with corresponding probabilities:

$$\begin{aligned}
(X = 0 \text{ and } Y = r) &\quad \text{with prob. } a_0 b_r, \\
(X = 1 \text{ and } Y = r-1) &\quad \text{with prob. } a_1 b_{r-1}, \\
(X = 2 \text{ and } Y = r-2) &\quad \text{with prob. } a_2 b_{r-2}, \\
&\quad \vdots \\
(X = r \text{ and } Y = 0) &\quad \text{with prob. } a_r b_0.
\end{aligned}$$

Hence the distribution of Z is given by:

$$C_r = P_r\{Z = r\} = a_0 b_r + a_1 b_{r-1} + a_2 b_{r-2} + \cdots + a_r b_0$$

$$C_r = \sum_{i=0}^{r} a_i b_{r-i}.$$

Then the new sequence $\{C_r\}$ which results from a combination of the two sequences $\{a_k\}$ and $\{b_j\}$ is called the convolution of $\{a_k\}$ and $\{b_j\}$, and is denoted by:

$$\{C_r\} = \{a_k\} * \{b_j\}.$$

Let $P_X(S)$, $P_Y(S)$, and $P_Z(S)$ be the p.g.f.'s of X, Y, and Z respectively, then from (1.9) it follows that:

$$\begin{aligned}
P_Z(S) &= E(S^{X+Y}) = E(S^X S^Y) \\
&= E(S^X)\, E(S^Y) = P_X(S) \cdot P_Y(S).
\end{aligned}$$

This is because of the independence of X and Y.

Where:

Step 1: Define generating functions

$$P_X(S) = \sum_{i=0}^{\infty} a_i S^i, \qquad P_Y(S) = \sum_{j=0}^{\infty} b_j S^j,$$

$$= \sum_{r=0}^{\infty} C_r S^r, \quad \text{where } C_r = P(Z = r).$$

Step 2: Express C_r as convolution

$$C_r = \sum_{i=0}^{r} P(X = i,\, Y = r - i)$$

$$= \sum_{i=0}^{r} a_i b_{r-i}.$$

Step 3: Substitute into $P_Z(S)$

$$P_Z(S) = \sum_{r=0}^{\infty} C_r S^r = \sum_{r=0}^{\infty} \sum_{i=0}^{r} a_i b_{r-i} S^r.$$

Step 4: Separate powers of S

$$= \sum_{r=0}^{\infty} \sum_{i=0}^{r} a_i b_{r-i} S^i S^{r-i} = \sum_{r=0}^{\infty} \sum_{i=0}^{r} (a_i S^i)(b_{r-i} S^{r-i}).$$

Step 5: Rearrange the double sum

$$= \sum_{i=0}^{\infty} a_i S^i \sum_{r=i}^{\infty} b_{r-i} S^{r-i}.$$

Step 6: Recognize the product of series

Let $j = r - i \Rightarrow r = i + j$.

$$P_Z(S) = \sum_{i=0}^{\infty} a_i S^i \sum_{j=0}^{\infty} b_j S^j$$

$$= P_X(S)\, P_Y(S).$$

Theorem (1):

The p.g.f. of the sum of two independent random variables X and Y is the product of the p.g.f.'s of X and Y, i.e.

$$P_Z(S) = P_X(S) \cdot P_Y(S). \tag{1.11}$$

The result also holds in the case of the sum S_n of (n) non-negative independent integer-valued r.v.'s $X_1, X_2, \ldots, X_n$, i.e.

$$P_{S_n}(S) = P_{X_1}(S) \cdot P_{X_2}(S) \cdots P_{X_n}(S) = \prod_{i=1}^{n} P_{X_i}(S). \tag{1.12}$$

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Theorem (2):

The sum $S_n = X_1+X_2+\cdots+X_n$ of fixed number (n) of independent and identically distributed (i.i.d.) r.v.'s X_i has p.g.f. as:

$$P_{S_n}(S) = P_{X_1}(S)\,P_{X_2}(S)\cdots P_{X_n}(S),$$
$$P_{S_n}(S) = [P_X(S)]^n. \tag{1.13}$$

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Example:

Let X_1 and X_2 be two independent Poisson variates with means (parameters) λ_1 and λ_2 respectively. Find the p.g.f. of the sum $Z = X_1 + X_2$ and the mean and variance of Z.

Solution: We know that the p.g.f of X_i is:

$$\begin{aligned}
P_{X_i}(S) &= \sum_{k=0}^{\infty} p_k S^k, \quad i = 1,2 \\
&= \sum_{k=0}^{\infty} \frac{e^{-\lambda_i}\lambda_i^k}{k!} S^k \\
&= e^{-\lambda_i} \sum_{k=0}^{\infty} \frac{(\lambda_i S)^k}{k!} \\
&= e^{-\lambda_i} e^{\lambda_i S} \\
&= e^{\lambda_i (S-1)}, \quad i = 1,2.
\end{aligned}$$

Then the p.g.f. of Z is:

$$P_Z(S) = \prod_{i=1}^{2} P_{X_i}(S) = P_{X_1}(S) P_{X_2}(S) = e^{\lambda_1(S-1)} e^{\lambda_2(S-1)} = e^{(\lambda_1+\lambda_2)(S-1)}.$$

Thus Z has a Poisson distribution with parameter $(\lambda_1+\lambda_2)$:

$$Z \sim \text{Poisson}(\lambda_1+\lambda_2).$$

The mean of Z is:

$$E(Z) = P_Z'(1) = (\lambda_1+\lambda_2)e^{(\lambda_1+\lambda_2)(S-1)}\big|_{S=1} = \lambda_1+\lambda_2.$$

The variance of Z is:

$$\text{var}(Z) = P_Z''(1) + P_Z'(1) - [P_Z'(1)]^2.$$

Now,

$$P_Z''(1) = (\lambda_1+\lambda_2)^2 e^{(\lambda_1+\lambda_2)(S-1)}\big|_{S=1} = (\lambda_1+\lambda_2)^2.$$

Hence:

$$\text{var}(Z) = (\lambda_1+\lambda_2)^2 + (\lambda_1+\lambda_2) - (\lambda_1+\lambda_2)^2 = \lambda_1+\lambda_2.$$

Thus, the variance of Z is $\lambda_1+\lambda_2$.