Examples of P.G.F. of Sum of Fixed Number of r.v.'s

Example :

If $X_1, X_2, \ldots, X_n$ be n independent Poisson distributions with parameters $\lambda_i,\ i = 1, 2, \ldots, n$, then the sum

$$S_n = \sum_{i=1}^{n} X_i$$

is a Poisson distribution with parameter $\sum_{i=1}^{n} \lambda_i$, and

$$E(S_n) = \text{var}(S_n) = \sum_{i=1}^{n} \lambda_i.$$

Solution: Since $X_i \sim \text{Poisson}(\lambda_i)$, then the p.g.f of X_i is:

$$P_{X_i}(S) = e^{\lambda_i (S-1)}, \quad i = 1, 2, \ldots, n.$$

Then the p.g.f. of $S_n = \sum_{i=1}^{n} X_i$ is:

$$\begin{aligned} P_{S_n}(S) &= \prod_{i=1}^{n} P_{X_i}(S) = P_{X_1}(S)\, P_{X_2}(S) \cdots P_{X_n}(S) \\ &= e^{\lambda_1 (S-1)} e^{\lambda_2 (S-1)} \cdots e^{\lambda_n (S-1)} \\ &= e^{\sum_{i=1}^{n} \lambda_i (S-1)} \\ &= e^{\lambda (S-1)}, \quad \lambda = \sum_{i=1}^{n} \lambda_i. \end{aligned}$$

Thus S_n is Poisson with parameter $\lambda = \sum_{i=1}^{n} \lambda_i$.
The mean is:

$$E(S_n) = P'_{S_n}(1) = \sum_{i=1}^{n} \lambda_i e^{\sum \lambda_i (S-1)} \big|_{S=1} = \sum_{i=1}^{n} \lambda_i.$$

The variance is:

$$\text{var}(S_n) = P''_{S_n}(1) + P'_{S_n}(1) - [P'_{S_n}(1)]^2.$$

Now,

$$P''_{S_n}(1) = \Big(\sum_{i=1}^{n} \lambda_i \Big)^2 e^{\sum \lambda_i (S-1)} \big|_{S=1} = \Big(\sum_{i=1}^{n} \lambda_i \Big)^2.$$

Then:

$$\text{var}(S_n) = \Big(\sum_{i=1}^{n} \lambda_i \Big)^2 + \sum_{i=1}^{n} \lambda_i - \Big(\sum_{i=1}^{n} \lambda_i \Big)^2 = \sum_{i=1}^{n} \lambda_i.$$

—

Example :

Let X_1 and X_2 be two independent random variables, each with the same Geometric distribution, with

$$p_k = q^k p, \quad k = 0, 1, 2, \ldots, n.$$

Find the p.g.f. of $Z = X_1 + X_2$, and the mean and variance of Z.

Solution: Since X_i is Geometric with p.g.f.:

$$P_{X_i}(S) = \frac{p}{1 - qS}, \quad i = 1, 2.$$

Then:

$$P_Z(S) = [P_X(S)]^n, \quad n = 2,$$

$$P_Z(S) = [P_X(S)]^2 = \left(\frac{p}{1 - qS}\right)^2 = \frac{p^2}{(1 - qS)^2}.$$

The mean and variance of Z are:

$$E(Z) = P'_Z(1) = -2p^2(1 - qS)^{-3}(-q)\Big|_{S=1} = 2p^2q(1 - q)^{-3}.$$

Since $q = 1 - p$, then

$$E(Z) = \frac{2q}{p}, \quad \text{the mean of } Z.$$

For the variance:

$$\text{var}(Z) = P''_Z(1) + P'_Z(1) - [P'_Z(1)]^2.$$

Now,

$$P''_Z(1) = -6p^2q(1 - qS)^{-4}(-q)\Big|_{S=1} = 6p^2q^2(1 - q)^{-4} = 6\frac{q^2}{p^2}.$$

Then:

$$\text{var}(Z) = \frac{6q^2}{p^2} + \frac{2q}{p} - \left(\frac{2q}{p}\right)^2.$$

$$= \frac{2q^2}{p^2} + \frac{2q}{p} = \frac{2q}{p}\left(\frac{q}{p} + 1\right) = \frac{2q}{p}\left(\frac{q + p}{p}\right) = \frac{2q}{p} \cdot \frac{1}{p} = \frac{2q}{p^2}.$$

Thus, $\text{var}(Z) = \dfrac{2q}{p^2}$.

—

Example :

If X_1 and X_2 are two independent r.v.'s with the same Geometric distribution and $p = \frac{2}{3},\ q = \frac{1}{3}$, find the p.g.f. of

$$Y = \sum_{i=1}^{2} X_i,$$

and find the mean and variance of Y.

Solution: Since X_i is Geometric with the same p.g.f.:

$$P_{X_i}(S) = \frac{p}{1-qS}, \quad i = 1, 2.$$

Then the p.g.f. of $Y = X_1 + X_2$ is:

$$P_Y(S) = [P_X(S)]^n, \quad n = 2,$$

$$P_Y(S) = \left(\frac{p}{1-qS}\right)^2 = \frac{p^2}{(1-qS)^2}.$$

The mean and variance of Y are:

$$E(Y) = P_Y'(S) = -2p^2(1-qS)^{-3}(-q)\Big|_{S=1} = \frac{2p^2q}{(1-q)^3}.$$

For $p = \frac{2}{3},\ q = \frac{1}{3}$,

$$E(Y) = \frac{2q}{p} = 1.$$

$$\text{var}(Y) = P_Y''(1) + P_Y'(1) - [P_Y'(1)]^2.$$

Now,

$$P_Y''(1) = -6p^2q(1-qS)^{-4}(-q)\Big|_{S=1} = 6\frac{q^2}{p^2}.$$

Then:

$$\text{var}(Y) = \frac{6q^2}{p^2} + 1 - (1)^2 = \frac{6q^2}{p^2}.$$

For $p = \frac{2}{3}, q = \frac{1}{3}$:

$$\text{var}(Y) = 6\left(\frac{1}{3}\right)^2\left(\frac{3}{2}\right)^2 = 6 \cdot \frac{1}{9} \cdot \frac{9}{4} = \frac{3}{2}.$$

Thus, $\text{var}(Y) = \frac{3}{2}$.