Lecture Seven of the Stochastic Processes (1) course.

Definition of Stochastic Processes

Introduction:

The theory of Stochastic Processes deals with systems with develops in time or space in accordance with probabilities laws. Many applications of Stochastic Processes occur in Physics, Engineering, Medicine and other fields.

Definition of Stochastic Processes

The Stochastic Processes is defining as a family of random variables $\{X(t),\ t \in T\}$ indexed by a parameter t or function of time space. Since a r.v. X of function of the possible outcomes S of an experiment it now becomes a function of both S and time t, $\{X(t, S);\ S \in \Omega;\ t \in T\}$, where the set T is called parameter space or index set of the process, and S is called state space or sample space.

Specification of Stochastic Processes:

There are two types of random variables $\{X_n\}$ define over all relevant values of n (discrete time) or $\{X(t)\}$ defined over all relevant values (continuous time). The set of possible values of an individual r.v. X_n or of the stochastic processes $\{X_n, n \geq 1\}$ or $\{X(t), t \in T\}$ is known as it states space S. The State space S of the process is the set of possible values of an individual $\{X_n\}$ or $\{X(t)\}$. The state space can be classified first as a one-dimensional or multi-dimensional, simple process usually being a one-dimensional secondly the state space is discrete if it contains a finite or countable infinity of points, and otherwise is continuous.

Then the one-dimensional stochastic processes can be classified according to the nature of state space S and parameter space T into one of the following types:

1. Both S and T are discrete.
2. S discrete and T continuous.
3. S continuous and T discrete.
4. Both S and T are continuous.

All four types may be represented by $\{X(t), t \in T\}$. If the time is discrete, we use $\{X_n, n = 0, 1, 2, \ldots\}$, and if the time is continuous, we use $\{X(t), t \in T\}$.

Example: (S and T are discrete)

Suppose that X_n is the outcomes of the n^{th} throw $(n \geq 1)$ of an experiment of throwing a die, then $\{X_n, n = 1, 2, \ldots\}$ is a family of r.v.'s such that for an index value of n, we get an independent random variable X_n.

Example: (S discrete and T continuous)

Consider a number of telephone calls received at a switchboard, and suppose that $X(t)$ is the r.v. which represents the number of incoming calls in an interval $(0, t)$ of duration t units. The number of calls with a fixed interval, say one unit of time (hour, day, week), is r.v. $X(t)$, and the family $\{X(t), t \in T\}$ constitute a stochastic process $T = (0, \infty)$.

Example: (S continuous and T discrete)

Suppose that X_n represented the average monthly of water in dam over any year, then $\{X_n, n = 0, 1, 2, \ldots\}$ is a stochastic process with continuous state space (rate of dam capacity), and discrete time $(1, 2, \ldots, 12)$.

Example: (S and T are continuous)

Suppose that $X(t)$ represented the maximum temperature at a particular place in $(0, t)$, then the set of possible values of $X(t)$ is continuous. Here we have a stochastic process with continuous time having continuous state space.

Mean and Variance of the process:

The mean function of a process is the function M defined by:

$$M_x(t) = E(X_t) = \int_{-\infty}^{\infty} X_t f_x(X_t; t) dX_t$$

And the variance function of the process $\{X(t), t \in T\}$ is the function K defined by:

$$K(t_1, t_2) = cov\{X(t_1), X(t_2)\} = K(t_2, t_1)$$

Where:

$$K(t_1, t_1) = var(t_1) = Var\{X(t_1)\}^2$$

Classification of Stochastic Processes:

Stationary Process: A stochastic process $\{X(t), t \in T\}$ is said to be a stationary process if the process does not change in time t under arbitrary translation of time parameter.

Strictly Stationary Process: A stochastic process $\{X(t), t \in T\}$ is said to be strictly stationary if the finite dimensional distribution are invariant under a time shift (stationary of order n), that is: for all $t_1, t_2, \ldots, t_n, n > 0$, then the process $\{X(t_1), X(t_2), \ldots, X(t_n)\}$ and $\{X(t_1+h), X(t_2+h), \ldots, X(t_n+h)\}$ have the same probability density function for all n, and for every set of time instants $(t_i \in T, i = 1, 2, \ldots, n)$ and any h, i.e.:

$$f_x\{X_1, X_2, \ldots, X_n; t_1, t_2, \ldots, t_n\} = f_x\{X_1, X_2, \ldots, X_n; t_1{+}h, t_2{+}h, \ldots, t_n{+}h\}$$

$$\begin{aligned} &P_r\{X(t_1) \leq X_1, X(t_2) \leq X_2, \ldots, X(t_n) \leq X_n\} \\ =&P_r\{X(t_1+h) \leq X_1, X(t_2+h) \leq X_2, \ldots, X(t_n+h) \leq X_n\} \end{aligned}$$

Then the first order distribution $(n = 1)$:

$$F_x(X; t) = F_x(X; t+h) = F_x(X)$$

and the mean and covariance of the process are existing, and have the same value, i.e.:

$$E\{X(t)\} = E\{X(t+h)\} = m$$

The mean must be constant and independent of t. And the covariance function exists, we shall assume that the mean is zero, then:

$$K(t_1, t_2) = cov\{X(t_1), X(t_2)\} = E\{X(t_1) \cdot X(t_2)\}$$

$$= E\{X(t_1+h) \cdot X(t_2+h)\}, \text{ for any } h,$$

then:

$$K(t_1, t_2) = E\{X(t_1 - t_2) \cdot X(0)\}$$

This shows that $K(t_1, t_2)$ is a function of the time difference $|t_1 - t_2|$.

Weakly Stationary: The process $\{X(t), t \in T\}$ is said to be weakly stationary process if its mean function and covariance function are invariant under a time shift:

- $m(t) = m,$ constant independent of t.
- $cov\{X(t), X(t+s)\} = var(s)$ is a function of s independent of t.

Gaussian Process: The processes $\{X(t), t \in T\}$ is called Gaussian process if all its finite dimensional distribution multivariate Normal distribution.

Non-Stationary Process: The process which is not stationary is called non-stationary process if the mean and variance depend on time t. That is:

- $E\{X(t)\} = m(t),$ is a function of t.
- $vov\{X(t)\} = K(t)$ is a function of t.

Independent Increments: For all $(t_1 < t_2 < \ldots < t_n)$, the process $\{X(t), t \in T\}$ is said to be independent increment if the increments:

$$X(t_2) - X(t_1),\, X(t_3) - X(t_2),\, \ldots,\, X(t_n) - X(t_{n-1}),$$

are independent.