$$f_X(x) = \int_0^{\infty} xe^{-x(y+1)}\,dy$$

$$= e^{-x}[-e^{-xy}]_0^{\infty} = e^{-x}, x \geq 0$$

and

$$f_Y(y) = \int_0^{\infty} xe^{-x(y+1)}\,dx$$

$$= \left.\frac{-xe^{-x(y+1)}}{(y+1)}\right]_0^{\infty} + \frac{1}{y+1}\int_0^{\infty} e^{-x(y+1)}\,dx$$

$$= 0 - \left.\frac{1}{(y+1)^2} e^{-x(y+1)}\right]_0^{\infty} = \frac{1}{(y+1)^2}, y \geq 0.$$

The marginal distribution of X is given by

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = \int_0^{x} e^{-x}\,dx$$

$$= 1 - e^{-x} \qquad x \geq 0$$

and the marginal distribution of Y is given by

$$F_Y(y) = \int_0^{y} \frac{1}{(y+1)^2}\,dy = -\left.\frac{1}{y+1}\right]_0^{y}$$

$$= \frac{y}{y+1}, y \geq 0$$

Example 7.2.4.

Let X and Y be jointly distributed with p.d.f. f(x,y) given by

$$f(x,y) = \begin{cases} \frac{1}{2} & , 0 < x < 2 , 0 < y < x \\ 0 & \text{otherwise} \end{cases}$$

1. Find the marginal p.d.f. 's of X and Y.
2. Find the joint probability distribution function.

Solution

1. The marginal p.d.f. of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_0^x \frac{1}{2}\,dy = \frac{x}{2} \qquad , 0 < x < 2$$

$$= 0 \qquad \text{otherwise} .$$

The marginal p.d.f. of Y is given by

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_y^2 \frac{1}{2}\,dx \text{ because}$$

$0 < y < x < 2$. Hence

$$f_Y(y) = \frac{1}{2}[2 - y] = 1 - \frac{y}{2} , 0 < y < 2 .$$

2. The joint distribution function is

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dy\,dx = \int_0^y\int_0^x \frac{1}{2}\,dx\,dy$$

$$= \frac{1}{2}xy \ , \ 0 < y < x < 2$$

7.3. Independent Random Variables:

In Chapter Four, we defined the independent events. Two events A and B are said to be independent if and only if P(AB)= P(A). P(B). From this definition we can define the independence of the two random variables. Let the event

$A = \{X \leq x\}$ and the event , $B = \{Y \leq y\}$, then

$$F(x,y) = P\{X \leq x, Y \leq y\} = P\{A \text{ and } B\}$$
$$= P\{A\}.\ P\{B\} \text{ if A and B are independent}$$
$$= F_X(x).\ F_Y(y)$$

so we have

Definition 7.3.1.

The joint distributed r.v. 's X and Y are said to be independent if and only if their joint distribution function F(x,y) can be written as the product of the marginal distribution functions $F_X(x)$ and $F_Y(y)$, i.e.

$$F(x,y) = F_X(x).F_Y(y) \quad \text{.........} \ (7.3.1)$$

If X and Y are continuous random variables, with joint p.d.f. f(x,y), then X and Y are independent if and only if

$$f(x,y) = f_X(x).f_Y(y) \quad \text{...........} \ (7.3.2)$$

In the same way, if X and Y are discrete random variables with joint p.m.f. P(x,y), then X and Y are independent if and only if

$$P(x,y) = P_X(x).P_Y(y) \quad \text{..........} \ (7.3.3)$$

for all x and y.

Example 7.3.1.

Let X and Y be two r.v. 's having the joint p.d.f.

$$f(x,y) = \begin{cases} 4e^{-2(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

The marginal density functions of X and Y are. respectively

$$f_X(x) = \int_0^\infty 4e^{-2(x+y)}\,dy = 2e^{-2x}, x > 0,$$

and

$$f_Y(y) = \int_0^\infty 4e^{-2(x+y)}\,dx = 2e^{-2y}, y > 0 \quad .$$

Then

$$f_X(x).f_Y(y) = (2e^{-2x})(2e^{-2y})$$
$$= 4e^{-2(x+y)} = f(x,y).$$

Therefore X and Y are independent r.v.'s.

Example 7.3.2.

Let X and Y be two r.v. 's with joint p.m.f. given in the table below:

Y = y / X=x	0	1	Total $P_X(x)$
0	4/9	2/9	6/9
1	2/9	1/9	3/9
Total $P_Y(y)$	6/9	3/9	1

We have

$$P_X(0).P_Y(0) = \frac{6}{9}.\frac{6}{9} = \frac{4}{9} = P(0,0)$$

$$P_X(1).P_Y(0) = \frac{3}{9}.\frac{6}{9} = \frac{2}{9} = P(1,0)$$

$$P_X(0).P_Y(1) = \frac{6}{9}.\frac{3}{9} = \frac{2}{9} = P(0,1)$$

and

$$P_X(1).P_Y(1) = \frac{3}{9}.\frac{3}{9} = \frac{1}{9} = P(1,1)$$

Then X and Y are independent.

7.4. Expectation of a Function of Two Random Variables

If X and Y are jointly r.v. 's with p.d.f f(x,y) (or with p.m.f. P(x,y), then the function Z = h (x,y) is also a r.v. which is a function of the two r.v. 's X and Y. We can define the expectation of the function Z in the same manner as in Proposition 6.4.1, so we have

$$E(Z) = E\{h(X,Y)\} = \Sigma\Sigma\, h(x,y)\,P(x,y) \quad \text{.......... (7.4.1)}$$
if X and Y are discrete r.v. 's

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\,f(x,y)\,dx\,dy \text{ if X and Y} \quad \text{.......... (7.4.2)}$$
are continuous r.v. 's

Example 7.4.1.

Two r.v. 's X and Y have the joint p.d.f.

$$f(x,y) = 3 - x - 3y\ ,\ 0 \le x \le 1\ ,\ 0 \le y \le 1\ .$$

Then

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x, y)\, dx\, dy$$

$$= \int_0^1\int_0^1 xy(3 - x - 3y)\, dy\, dx$$

$$= \int_0^1 x\left[\frac{3y^2}{2} - \frac{xy^2}{2} - \frac{3y^3}{3}\right]_0^1 dx$$

$$= \int_0^1 x\left(\frac{3}{2} - \frac{1}{2}x - 1\right) dx$$

$$= \frac{3x^2}{4} - \frac{x^3}{6} - \frac{x^2}{2}\Bigg]_0^1 = \frac{3}{4} - \frac{1}{6} - \frac{1}{2}$$

$$= \frac{1}{12}.$$

Theorem 7.4.1

Let X and Y be two r.v.'s and if E(X) and E(Y) exist, then

$$E(X + Y) = E(X) + E(Y) \qquad \dots (7{\cdot}4{\cdot}3)$$

$$E(XY) = E(X).E(Y) \qquad \dots (7{\cdot}4{\cdot}4)$$

if X and Y are independent r. v. s.

• ***Proof:*** –

Let X and Y be discrete r. v.'s with joint p. m. f. $P(x, y)$, then by (7·4·1) we have

i. $E(X+Y) = \sum_x \sum_y (x+y)\, P(x,y)$

$$= \sum_x \sum_y x\, P(x,y) + \sum_y \sum_x y\, P(x,y)$$

$$= \sum_x x\, P_X(x) + \sum_y y\, P_Y(y)$$

$$= E(X) + E(Y).$$

ii. $E(XY) = \sum_x \sum_y xy\, P(x,y)$

$$= \sum \sum xy\, P_X(x) P_Y(y) \text{ since X and Y are independent}$$

$$= \sum_x x\, P_X(x) . \sum_y y\, P_Y(y) = E(X)\, E(Y)$$

Now, let X and Y be continuous r.v. 's with joint p.d.f. f(x,y), then by (7.4.2) we have

i. $E(X+Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x+y)\, f(x,y)\, dx\, dy$

$$\int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} f(x,y)\, dy\, dx + \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x\, f_X(x)\, dx + \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

$$= E(X) + E(Y).$$

ii. $E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f_X(x) . f_Y(y)\, dx\, dy \text{ , since X and Y are independent}$$

$$= \int_{-\infty}^{\infty} x f_X(x)\,dx \int_{-\infty}^{\infty} y f_Y\,dy = E(X)E(Y).$$

Remarks.

i. As in Theorem 7.4.1, we can also show that for any r.v.'s X and Y and any constants a and b,

$$E[aX + bY] = aE(X) + bE(Y) \quad \ldots\ldots\ldots\ldots\ldots (7.4.5)$$

ii. Theorem 7.4.1 can be extended to n random variables X_1, X_2, ... , X_n, that is

a· $$E(X_1 + X_2 + \ldots + X_n) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) \ldots (7{\cdot}4{\cdot}6)$$

b· $$E(X_1X_2 \ldots X_n) = E\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} [E(X_i)] \quad \ldots (7{\cdot}4{\cdot}7)$$

if $X_1, X_2, \ldots, X_n$ are independent r. v.'s

iii· If $X_1, X_2, \ldots, X_n$ are n r. v.' and for $a_1, a_2, \ldots, a_n$ are constants, we have

$$E\left[\sum_{i=1}^{n} a_iX_i\right] = \sum_{i=1}^{n} a_i E(X_i) \quad \ldots (7{\cdot}4{\cdot}8)$$

Definition 7.4.1.

Let X and Y be two r.v. 's, the *covariance* function is denoted by Cov (X,Y), and is defined as

$$\begin{aligned} Cov(X,Y) &= E[\{X-E(X)\}\{Y-E(Y)\}] \\ &= E(XY) - E(X)E(Y) \quad \ldots\ldots. (7.4.9) \end{aligned}$$

If X and Y are independent random variables, then they are uncorrelated, i.e.

$$Cov(X,Y) = 0$$

because $E(XY) = E(X)\,E(Y)$ by (7.4.4). But the converse is not true.

Definition 7.4.2.

The *correlation coefficient* of X and Y, is denoted by $\rho(X, Y)$ and is defined by

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X).Var(Y)}} \quad \ldots (7{\cdot}4{\cdot}10)$$

$$= \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

where σ_X and σ_Y are the standard deviations of X and Y, respectively.

We have

$$|\rho(X,Y)| \leq 1$$

Lemma- 7.4.1. Cauchy- Schwartz Inequality

If X and Y are random variables having a finite second moments, then

$$[E(XY)]^2 \leq E(X^2)E(Y^2)$$

with equality if and only if $P[aX = bY] = 1$ for some constants a and b, at least one of which is non-zero.

Proof:

Let $Z = aX - bY$, then $Z^2 \geq 0$. Thus
$E(Z^2) = a^2E(X^2) - 2ab\,E(XY) + b^2E(Y^2) \geq 0$

It is a quadratic expression in a with at most one real roots if its discriminant must be nonpositive. That is
$4a^2b^2(E(XY))^2 - 4a^2b^2E(X^2)E(Y^2) \leq 0$

If $b \neq 0$, then
$[E(XY)]^2 \leq E(X^2)E(Y^2)$.

The discriminant is zero if and only if the quadratic expression has a real root. This occurs if and only if

$E(aX - bY)^2 = 0$ for some a,b.

Therefore, $P(aX - bY = 0) = 1$.

Theorem 7.4.2.

Let X and Y be two r.v. 's , then for any constants a and b, we have

$$\begin{aligned} Var[aX + bY] &= a^2 Var(X) + b^2 Var(Y) \\ &+ 2ab\, Cov(X, Y) \end{aligned} \quad \dots (7{\cdot}4{\cdot}11)$$

Proof:

We have

$$\begin{aligned} Var[aX + bY] &= E[aX + bY]^2 - [E(aX + bY)]^2 \\ &= E[aX + bY]^2 - [aEX + bEY]^2 \\ &= E[a^2X^2 + 2abXY + b^2Y^2] - a^2(EX)^2 \\ &- 2abEXEY - b^2(EY)^2 \\ &= a^2E(X^2) - a^2(E(X))^2 + 2abE(XY) \\ &- 2ab\,E(X)E(Y) + b^2E(Y^2) - b^2(E(Y))^2 \\ &= a^2[E(X^2) - (E(X))^2] + b^2[E(Y^2) \\ &- (E(Y))^2] + 2ab[E(XY) - E(X)E(Y)] \\ &= a^2\, Var(X) + b^2\, Var(Y) + 2ab\, Cov(X, Y). \end{aligned}$$

If X and Y are independent, then $Cov(X,Y) = 0$ and hence

$$Var[aX + bY] = a^2\, Var(X) + b^2 Var(Y) \quad \dots (7{\cdot}4{\cdot}12)$$

Theorem 7.4.2 can be extended to n r.v. 's. $X_1, X_2, \dots, X_n$, that is

$$Var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2\, Var(X_i) + 2\sum_{i} \sum_{\neq j} a_i a_j Cov(X_i, X_j)$$

$$\dots (7{\cdot}4{\cdot}13)$$

where a_i, $i=1,2, \dots, n$ are constants.