Example 7.4.1.

Let X and Y be two independent r.v. 's having binomial distributions with parameters (n, p_1) and (m, p_2), respectively Find

i. $E(5X_1 + 4X_2)$ and $Var(5X_1 + 4X_2)$

ii. $E(3X_1 + X_2)$ and $Var(3X_1 - X_2)$

Solution

Since X_1 has a binomial distribution with parameters n and p_1, then

$$E(X_1) = np_1 \text{ and } Var(X_1) = np_1 q_1 .$$

Also, we have

$$E(X_2) = mp_2 \text{ and } Var(X_2) = m p_2 q_2.$$

Therefore,

i. $E(5X_1 + 4X_2) = 5E(X_1) + 4E(X_2)$

$$= 5np_1 + 4mp_2$$

and

$Var(5X_1 + 4X_2) = 25\, Var(X_1) + 16\, Var(X_2)$ because X_1 and X_2 are independent, $Var(5X_1 + 4X_2) = 25\, np_1q_1 + 16\, m p_2 q_2$.

ii. $E(3X_1 + X_2) = 3E(X_1) + E(X_2)$

$$= 3np_1 + mp_2$$

and

$$Var(3X_1 - X_2) = 9\, Var(X_1) + Var(X_2)$$

$$= 9\, np_1q_1 + m p_2q_2$$

Example 7.4.2.

Let X and Y be two random variables. with joint p.m.f. given below

Y=y / X=x	0	1	2	$P_X(x)$
0	0.05	0.10	0.02	0.17
1	0.08	0.15	0.05	0.28
2	0.20	0.10	0.05	0.35
3	0.10	0.05	0.05	0.20
P_y (y)	0.43	0.40	0.17	1

Compute E (X), E (Y), Cov (X,Y) and ρ(X,Y).

Solution

$$E(X) = \Sigma x P_X(x)$$
$$= 0(0.17) + 1(0.28) + 2(0.35) + 3(0.20)$$
$$= 0.28 + 0.70 + 0.60 = 1.58$$

$$E(Y) = \Sigma\ y\, p_Y(y)$$
$$= 0(0.43) + 1(0.40) + 2(0.17)$$
$$= 0.40 + 0.34 = 0.74$$

$$E(XY) = \Sigma\Sigma\ xy\, P(x,y)$$
$$= 0(0)(0.05) + 0(1)(0.10) + 0(2)(0.02)$$
$$+ 1(0)(0.08) + 1(1)(0.15) + 1(2)(0.05)$$
$$+ 2(0)(0.20) + 2(1)(0.10) + 2(2)(0.05)$$
$$+ 3(0)(0.10) + 3(1)(0.05) + 3(2)(0.05)$$
$$= 0.15 + 0.10 + 0.20 + 0.20 + 0.15 + 0.30$$
$$= 1.10.$$

Therefore

$$Cov(X,Y) = E(XY) - EX\,EY$$
$$= 1.10 - (1.58)(0.74)$$
$$= 1.10 - 1.192 = -0.0692.$$

To find $\rho(X, Y)$, we first calculate

$$Var(X) = E(X)^2 - (E(X))^2$$

$$E(X^2) = (0)^2\ (0.17) + (1)^2\ (0.28) + (2)^2\ (0.35) + (3)^2\ (0.20)$$
$$= 0.28 + 1.40 + 1.80 = 3.48$$
$$Var\ (X) = 3.48 - (1.58)^2$$
$$= 3.48 - 2.4964 = 0.9836 \Longrightarrow \sigma_X = 0.9917$$

$$E(Y^2) = (0)^2 (0{\cdot}43) + (1)^2 (0{\cdot}40) + (2)^2 (0{\cdot}17)$$
$$= 0{\cdot}40 + 0{\cdot}68 = 1{\cdot}08$$
$$Var(Y) = 1{\cdot}08 - (0{\cdot}74)^2$$
$$= 1{\cdot}08 - 0{\cdot}5476$$
$$= 0{\cdot}5324 \Longrightarrow \sigma_Y = 0{\cdot}7296.$$

Therefore

$$\rho(X,Y) = \frac{Cor(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \frac{-0{\cdot}0692}{(0{\cdot}9917)(0{\cdot}7296)}$$

$$= \frac{-0{\cdot}0692}{0{\cdot}7235} = -0{\cdot}0956.$$

Example 7.4.3.

Consider Example 7.4.1. We have been shown that $E(XY) = 1/12$

To find the correlation coefficient, we need to find EX, EY, Var, (X) and Var (Y). We have

$$E(X) = \int_{-\infty}^{\infty} x\, f_x(x)\, dx$$

where

$$f_x(x) = \int f(x,y)\, dy$$

$$= \int_0^1 (3 - x - 3y)\, dy$$

$$= 3y - xy - \frac{3y^2}{2}\Bigg]_0^1 = \frac{3}{2} - x, \; 0 < x < 1$$

and

$$f_Y(y) = \int f(x,y)\,dx$$

$$= \int_0^1 (3 - x - 3y)\,dx$$

$$= 3x - \frac{x^2}{2} - 3yx \Big]_0^1 = \frac{5}{2} - 3y, 0 < y < 1$$

Hence

$$EX = \int_0^1 x\left(\frac{3}{2} - x\right)dx = \frac{3}{4}x^2 - \frac{x^3}{3}\Big]_0^1$$

$$= 5/12,$$

and

$$EX^2 = \int_0^1 x^2\left(\frac{3}{2} - x\right)dx = \frac{x^3}{2} - \frac{x^3}{4}\Big]_0^1 = \frac{1}{4}$$

then $Var(X) = EX^2 - (EX)^2 = \frac{1}{4} - \left(\frac{5}{12}\right)^2 = 11/144$

In the same way we find EY and Var (Y).

$$EY = \int_0^1 y\left(\frac{5}{2} - 3y\right)dy = \frac{5y^2}{4} - y^3\Big]_0^1 = \frac{1}{4},$$

and

$$EY^2 = \int_0^1 y^2\left(\frac{5}{2} - 3y\right)dy = \frac{5y^3}{6} - \frac{3y^4}{.4}\Big]_0^1$$

$$= 1/12,$$

then

$$\text{Var}(Y) = EY^2 - (EY)^2 = \frac{1}{12} - \left(\frac{1}{4}\right)^2 = 1/48.$$

Therefore, the correlation coefficient between the two r.v. X, and Y is

$$\rho(X,Y) = \frac{\text{Cor}(X,Y)}{\sigma_X \sigma_Y} = \frac{EXY - EX\,EY}{\sigma_X \sigma_Y}$$

$$= \frac{\frac{1}{12} - \frac{5}{12} \cdot \frac{1}{4}}{\sqrt{11/144}\ \sqrt{1/48}} = \frac{-1/48}{(0{\cdot}276)(0{\cdot}144)}$$

$$= \frac{-0{\cdot}0208}{0{\cdot}03974} = -0{\cdot}523$$

7.5 Conditional Probability Functions

In Chapter Four, we d[illegible]ditional probability of the event A given the [illegible]

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}, \quad P(B) > 0.$$

From this definition we can define the conditional probability functions.

If X and Y are jointly discrete r.v.'s with the joint p.m.f. P(x,y), the *conditional probability mass function* of X, given Y=y, is denoted by $P_{X|Y}(x|y)$, and is defined for all y such that $P_Y(y) > 0$ by

$$P_{X|Y}(x \mid y) = P(X = x \mid Y = y) = \frac{P(x,y)}{P_Y(y)} \qquad \ldots (7{\cdot}5{\cdot}1)$$

In the same way we defined the *conditional probability mass function* of Y, given X=x which is denoted by $P_{Y|X}$ (y|x), and is defined for all x such that $P_{Y|X} > 0$ by

$$P_{Y|X}(y \mid x) = P(Y = y \mid X = x) = \frac{P(x, y)}{P_X(X)} \qquad \ldots (7{\cdot}5{\cdot}2)$$

If X and Y are jointly continuous random variables with joint p. d. f. f(x, y), the conditional probablitiy density function of X given Y=y, is denoted by $f_{X|Y}$ (x|y), and is defined for all y such that $f_Y(y) > 0$, as

$$f_{X\,Y}(x \mid y) = \frac{f(x, y)}{f_Y(y)} \qquad \ldots (7{\cdot}5{\cdot}3)$$

The conditional probability density function of Y given X=x , is denoted by $f_{Y/X}$ (y|x), and is defined for all x, such that

$f_X(x) > 0$., by

$$f_{Y|X}(y \mid x) = \frac{f(x, y)}{f_X(x)} \qquad \ldots (7{\cdot}5{\cdot}4)$$

Example 7.5.1

Given the joint p.m.f. of X and Y

$$P(x, y) = \frac{1}{30}(x + y) \text{ for } x = 0, 1, 2, 3, \text{ and } y = 0, 1, 2.$$

Show that

a. the marginal p.m.f. of X is

$$P_X(x) = \frac{1}{10}(x + 1), \text{ for } x = 0, 1, 2, 3$$

b. the marginal p.m.f. of Y is

$$P_Y(y) = \frac{2y+3}{15}, \text{ for } y = 0, 1, 2.$$

c. Find $P_{X|Y}(x|y)$, and use it to evaluate $P_{X|Y}(2|2)$, $P_{X|Y}(0|1)$, and $P_{X/Y}(3|0)$.

d. Find $P_{Y/X}(y|x)$, and use it to evaluate $P_{Y|X}(2|2)$, $P_{Y|X}(1|3)$, and $P_{Y|X}(0|0)$

Solution: we have

a. $$P_X(x) = \sum_y P(x, y)$$

$$= \frac{1}{30} \sum_{y=0}^{2} (x+y) = \frac{(x+0)+(x+1)+(x+2)}{30}$$

$$\frac{3x+3}{30} = \frac{x+1}{10}, x = 0,1,2,3.$$

b. $$P_Y(y) = \sum_x P(x, y)$$

$$= \frac{1}{30} \sum_{x=0}^{3} (x+y) = \frac{(y+0)+(y+1)+(y+2)+(y+3)}{30}$$

$$= \frac{4y+6}{30} = \frac{2y+3}{15}, y = 0, 1, 2.$$

c. $$P_{X|Y}(x|y) = \frac{P(x, y)}{P_Y(y)} = = \frac{(x+y)/30}{(2y+3)/15}$$

$$= \frac{x+y}{4y+6}, x = 0, 1, 2, 3.$$

Therefore,

$$P_{X|Y}(2/2) = \frac{2+2}{8+6} = \frac{2}{7},$$

$$P_{X|Y}(0|1) = \frac{0+1}{4+6} = \frac{1}{10},$$

and

$$P_{X|Y}(3|0) = \frac{3+0}{6} = 1/2.$$

d. $$P_{Y|X}(y|x) = \frac{P(x,y)}{P_X(x)} = \frac{(x+y)/30}{(x+1)/10} = \frac{x+y}{3x+3},$$

$y = 0, 1, 2.$

Therefore,

$$P_{Y|X}(2|2) = \frac{2+2}{6+3} = 4/9,$$

$$P_{Y|X}(1|3) = \frac{3+1}{9+3} = 1/3,$$

$$P_{Y|X}(0|0) = \frac{0+0}{0+3} = 0.$$

Example 7.5.2.

If two r.v.'s X and Y have the joint probability density

$$f(x,y) = \begin{cases} \frac{2}{3}(x+2y) \text{ for } 0 < x < 1, \quad 0 < y < 1 \\ 0 \text{ other wise} \end{cases}$$

Find the conditional probability density functions of X and Y.

Solution

The marginal p.d.f.'s of X and Y are respectively,

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

$$= \int_0^1 \frac{2}{3}(x + 2y)\, dy = \begin{cases} \frac{2}{3}(x+1), & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

$$= \int_0^1 \frac{2}{3}(x + 2y)\, dx = \begin{cases} \frac{1 + 4y}{3}, & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

The conditional P. d. f of X given Y = y is

$$f_{X|Y}(x \mid y) = \frac{f(x, y)}{f_Y(y)} = \frac{\frac{2}{3}(x + 2y)}{\frac{1 + 4y}{3}}$$

$$= \begin{cases} \frac{2x + 4y}{1 + 4y} & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

and the conditional p. d. f. of Y given X = x is

$$f_{Y|X}(y \mid x) = \frac{f(x, y)}{f_X(x)} = \frac{\frac{2}{3}(x + 2y)}{\frac{2}{3}(x + 1)}$$

$$= \begin{cases} \frac{x + 2y}{x + 1}, & 0 < y < 1. \\ 0 & \text{otherwise.} \end{cases}$$

7.6. Conditional Expectation and Variance

For two discrete r.v.'s X and Y with the joint p.m.f. P(x,y), the *conditional expectation* of X, given Y=y, is denoted by E (X |Y=y), and is defined for all y such that $P_Y(y) > 0$ as

$$E(X \mid Y = y) = \sum_x x\, P(X = x \mid Y = y) = \Sigma x P_{X|Y}(x|y),$$

$$= \frac{\sum_x x\, P(x, y)}{P_Y(y)} \qquad \dots (7{\cdot}6{\cdot}1)$$

The *conditional variance* of X, given Y=y, is denoted by Var (X |Y=y), and is defined for all y such that $P_Y(y) > 0$ as

$$\text{Var}\ (X \mid Y = y) = E(X^2 \mid Y = y) - [E(X \mid Y = y)]^2 \qquad \dots (7{\cdot}6{\cdot}2)$$

where

$$E(X^2 \mid Y = y) = \sum_x x^2\, P(X = x \mid Y = y)$$

$$= \sum_x x^2\, P_{X|Y}(x \mid y) = \frac{\sum_x x^2\, P(x, y)}{P_Y(y)}$$

$$\dots (7{\cdot}6{\cdot}3)$$

In the same way, we define the conditional expectation and variance of Y , given X=x for all x such that $P_X(x) > 0$ as

$$E(Y \mid X = x) = \sum_y y\, P(Y = y \mid X = x) = \sum_y y\, P_{Y|X}(y \mid x)$$

$$= \frac{\sum_y y\, P(x, y)}{P_X(x)} \qquad \dots (7{\cdot}6{\cdot}4)$$