4

Chapter Four

Axiomatic Approach of Probability

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Axiomatic Approach of Probability Theory

4.1 Probability Defined on Events

In this Chapter we develop a mathematical model of an experiment, which is defined mathematically. by three things.

1. Specifying the sample space S on which probability statements are made;
2. defining the events of interest; and
3. defining a numerical measure for a probability statement, which is assigned to the events.

The sample space S may be a finite or countably infinite set, then S is discrete. Some experiments have an uncountably infinite sample space which is called *continuous*.

The events may be either discrete or continuous, as with the sample space.

To each event E defined on a sample space S We shall assign a non negative real number which is called *probability* of E., denoted by P (E)

4.2. Axioms of Probability

In 1933, Kolmogorov gave the following definition of probability :

Let S be a sample space associated with a statistical experiment. Let F be a σ – field, which is a collection of subsets of S. A set function P defined on F is Called a *probability measure* (or simply probability) if it satisfies the following axioms:

Axiom 1.

$o \leq P(A) \leq 1$, for all $A \in F$,

Axiom 2.

$P(S) = 1$

Axiom 3.

For any sequence of mutually exclusive events $A_1, A_2, \ldots$ of F, that is, events for which $A_i A_j = \phi$ for $i \neq j$,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

Axiom 3. states that for any sequence of mutually exclusive events, the probability of at least one of these events occurring is just the sum of their respective probabilities. This property is called *countable additivity*

Definition 4.2.1.

The triple (S, F, P) is called a *probability space,* where S is the sample space, F is a σ - field of subsets of S and P a probability measure on F.

The probability assigned to S is 1, by Axiom 2.

Sometimes other events $A \in F$ with $P(A) = 1$ will happen. If a statement holds for all points in A with $P(A) = 1$, then it is customary to say that the statement is true almost surely.

Proposition 4.2.1.

$P(\phi) = 0$

Proof.

Let $A_n = \phi$ for n = 1.2, ... Then $\bigcup_{n=1}^{\infty} A_n = \phi$. Then by Axiom 3, we have

$$P(\phi) = P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(\phi)$$

This is true only when $P(\phi) = 0$.

Proposition 4.2.2.

If $A_1, A_2, \ldots, A_n$ are mutually exclusive events, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

Proof :

Let $A_i = \phi$ for $i = n+1, n+2, \ldots$. By Axiom (3) we have

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{n} P(A_i)$$

$$+ P(\phi) + P(\phi) + \cdots = \sum_{i=1}^{n} P(A_i).$$

Theorem 4.2.1.

1. For any event $A \in F$, $P(\text{not } A) = P(A') = 1 - P(A)$.
2. For any two events $A \in F$ and $B \in F$, If $A \subset B$, then $P(A) \leq P(B)$.

Proof:

(1) We have $S = A \cup A'$, but A and A' are disjoint, then $P(S) = P(A) + P(A')$ by Proposition 4.2.2. By Axiom (2), we have $P(S) = 1$. So
$P(A) + P(A') = 1$
or $P(A') = 1 - P(A)$.

(2) we have $B = A \cup (A'B)$ (see Fig 4.1) union of mutually exclusive events.
Then by Proposition 4.2.2, we have

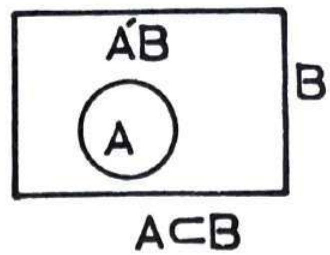

Fig. 4.1

$P(B) = P(A) + P(A'B)$ ..... (4,2.1).
The result follows because $P(A'B) \geq 0$.

From Equation (4.2 1) we have
$P(A'B) = P(B/A) = P(B) - P(A)$, if $A \subset B$
In general, For any two events A and B, We have
$P(B/A) = P(B) - P(AB)$ .... (4.2.2)

Theorem 4.2.2.

For any two events $A \in F$ and $B \in F$ we have
$P(A \cup B) = P(A \text{ or } B) = P(A) + P(B) - P(AB)$.

Proof:

From Fig 4.2, we have
$A \cup B = (A/B) \cap (A\ B) \cup (B/A)$ which are unions of mutually exclusive events. Then by Proposition 4.2.2, we have

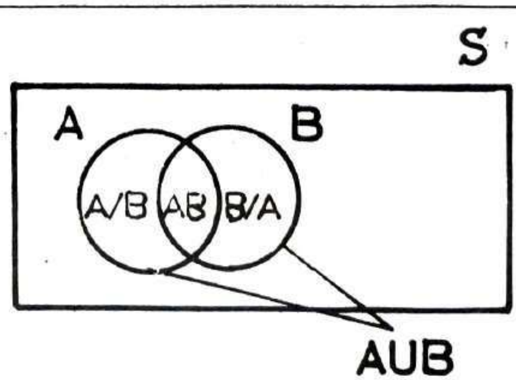

Fig 2·4

$P(A \cup B) = P(A/B) + P(AB) + P(B/A)$

By Equation (4.2.2), we have

$P(A/B) = P(A) - P(AB)$

and

$P(B/A) = P(B) - P(AB)$

Therefore.

$P(A \cup B) = P(A) - P(AB) + P(AB) + P(B) - P(AB)$

so the result follows.

Example 4.2.1

Two fair coins are tossed once. Let A be the event that the first coin falls heads, and B be the event that the second coin falls heads then

$A = \{HT. HH\}$

and

$B = \{TH, HH\}$

By Theorem 4.2.2 we have $P(A \cup B)$, the probability that either the first or the second coin falls heads, is given by

$$P(A \cup B) = P(A) + P(B) - P(AB)$$

$$P(AB) = P(\{HH\}) = \frac{1}{4}$$

$$\text{so } P(A \cup B) = \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}$$

Example 4.2.2

A committee of 4 persons are chosen at random from a class containing 7 girls and 9 boys. What is the probability that the committee will contain

i. 3 boys ;
ii. at least one girl?

Solution

i. $P(3 \text{ boys}) = P(3 \text{ boys and } 1 \text{ girl}) = \dfrac{h}{n}$

where $h = C(9,3) \cdot C(7,1) = 84\ (7) = 588$
and

$n = C(16,4) = 1820$

$\therefore\ P(3 \text{ boys}) = \dfrac{588}{1820} = \dfrac{21}{65}$

ii. $P(\text{at least one girl}) = P(1 \text{ girl}) + P(2 \text{ girls}) + P(3 \text{ girls}) + P(4 \text{ girls})$

$P(1 \text{ girl}) = P(1 \text{ girl and } 3 \text{ boys}) = \dfrac{C(9,3)\,C(7,1)}{C(16,4)} = \dfrac{21}{65}$,

$P(2 \text{ girls}) = P(2 \text{ girls and } 2 \text{ boys}) = \dfrac{C(9,2)\,C(7,2)}{C(16,4)} = \dfrac{27}{65}$,

$P(3 \text{ girls}) = P(3 \text{ girls and } 1\text{boy}) = \dfrac{C(9,1)\,C(7,3)}{C(16,4)} = \dfrac{9}{52}$,

and

$P(4 \text{ girls}) = \dfrac{C(7,4)}{C(16,4)} = \dfrac{1}{52}$,

therefore

$P(\text{at least one girl}) = \dfrac{21}{65} + \dfrac{27}{65} + \dfrac{9}{52} + \dfrac{1}{52} = \dfrac{121}{130}$

Another method:-

We have

$P(\text{no one}) + P(1 \text{ girl}) + P(2 \text{ girls}) + P(3 \text{ girls}) + P(4 \text{ girls}) = 1$

or

$P(\text{no one}) + P(\text{at least one girl}) = 1$

therefore

$P(\text{at least one girl}) = 1 - P(\text{no girl})$

But

$$P(\text{no girl}) = P(4 \text{ boys}) = \frac{C(9,4)}{C(16,4)} = \frac{126}{1820}$$

$$\therefore \; P(\text{at least one girl}) = 1 - \frac{126}{1820} = 1 - \frac{9}{130} = \frac{121}{130}$$

Theorem 4.2.2 can be extended to find the probability that any one of the three events A,B or C occurs :

$$P(A,B \text{ or } C) = P(A \cup B \cup C) = P(A)+P(B)+P(C) - P(AB)-P(AC) -P(BC) + P(ABC). \quad \ldots\ldots (4.2.3)$$

Example 4.2.3

Out of 120 students, 45 are studying French, 30 are studing Spanish, 25 are studying Italian, 14 are studying French and Spanish; 10 are studying Spanish and Italian, 8 are studying French and Italian, and 6 are studying the three languages. If a student is chosen at random, find the probability that the student.

i. is studying French or Spainsh,
ii. is studying Spanish or Italian,
iii. is studying one of the three languages,
iv. is not studying any one of the three languages.

Solution

Fig 4.3 shows how 120 students are classified according to whether or not the student is studying one or more of the languages.

Let
A = {the student is studying French}
B= {the student is studying Spanish,

and
C= {the student is studying Italian}.

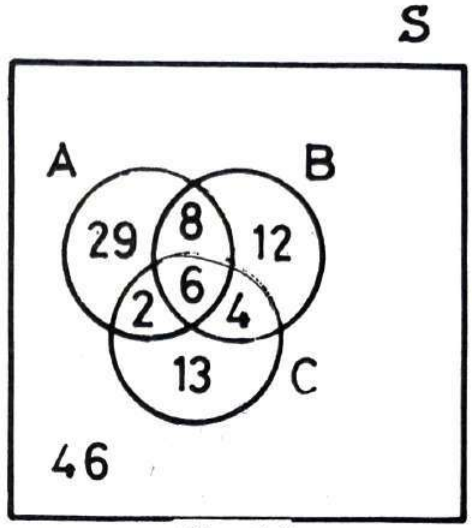

Fig. 4.3

i. P(A or B) = P(A⋃B) = P(A) +P(B) - P(AB)

$$= \frac{45}{120} + \frac{30}{120} - \frac{14}{120} = \frac{61}{120}$$

ii. P(B or C) = P(B) +P(C) - P(BC)

$$= \frac{30}{120} + \frac{25}{120} - \frac{10}{120} = \frac{45}{120}$$

iii. P(A,B or C) = P(A⋃B⋃C) = P(A) +P(B) +P(C) - P(AB) - P(AC) - P(BC) +P(ABC)

$$= \frac{45}{120} + \frac{30}{120} + \frac{25}{120} - \frac{14}{120} - \frac{10}{120} - \frac{8}{120}$$

$$+ \frac{6}{120} = \frac{74}{120}.$$

iv. P(A⋃B⋃C)′= 1-P(A⋃B⋃C)

$$= 1 - \frac{74}{120} = \frac{46}{120}.$$